Potenzsummen

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Potenzsummen

(0) $$\sum_{i=0}^{n} i^0 = 1n^1 + 1n^0$$

(1) $$\sum_{i=1}^{n} i^1 = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n^1 + 0n^0$$

(2) $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n^1 + 0n^0$$

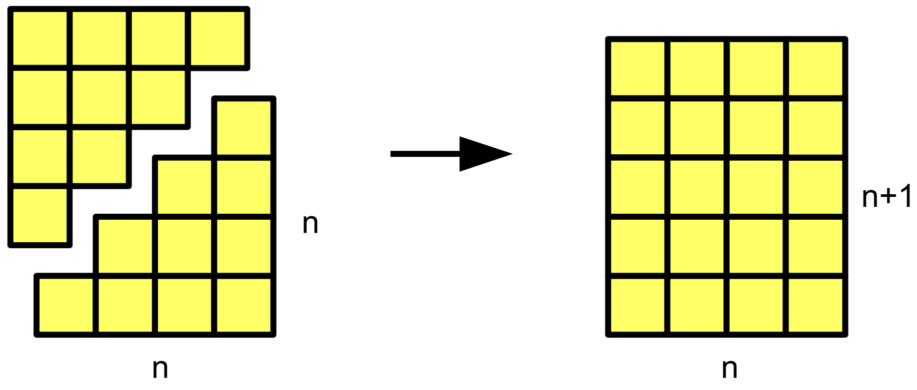
(3) $$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 + 0n^1 + 0n^0$$

(4) $$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3 + 0n^2-\frac{1}{30}n^1+0n^0$$

usw.

Beweis:

(1)

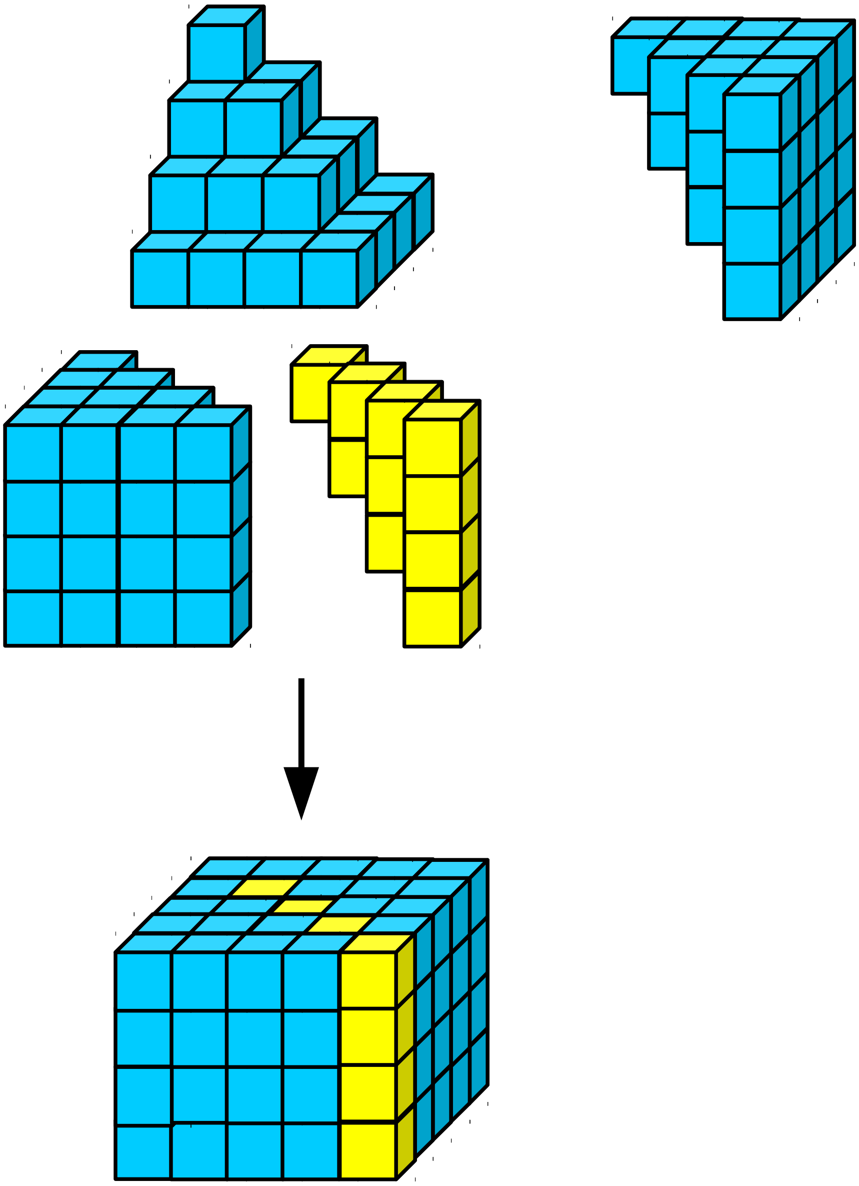


$$2\sum_{k=1}^{n} i \;=\; n(n+1)$$

$$\boxed{\sum_{k=1}^{n} i \;=\; \frac{n(n+1)}{2}}$$

Beweis:

(2)



$$3\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = (n+1)(n+1)n$$

$$3\sum_{i=1}^{n} i^2 + \frac{n(n+1)}{2} = (n+1)(n+1)n$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}\left((n+1)(n+1)n - \frac{n(n+1)}{2}\right)$$

$$\sum_{i=1}^{n} i^2 = \frac{2(n+1)(n+1)n - n(n+1)}{6}$$

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

Beweis:

(3)

$$\sum_{i=0}^{n} (i+1)^4 = \sum_{i=0}^{n} i^4+4i^3+6i^2+4i+1$$

$$\sum_{i=1}^{n} i^4 + (n+1)^4 = \sum_{i=0}^{n} i^4 + 4\sum_{i=0}^{n} i^3 + 6\sum_{i=0}^{n} i^2 + 4\sum_{i=0}^{n} i + \sum_{i=0}^{n} 1$$

$$\sum_{i=1}^{n} i^4 + (n+1)^4 = \sum_{i=1}^{n} i^4 + 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$$

$$(n+1)^4 = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$$

$$4\sum_{i=1}^{n} i^3 = (n+1)^4 - 6\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i - \sum_{i=0}^{n} 1$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left((n+1)^4 - 6\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i - \sum_{i=0}^{n} 1\right)$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left((n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - (n+1)\right)$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left(n^4+4n^3+6n^2+4n+1 - 2n^3-3n^2-n - 2n^2-2n - n-1\right)$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left(n^4+2n^3+n^2\right)$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left(n^2+n\right)^2$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left(n^2(n+1)^2\right)$$

$$\boxed{\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}}$$

Beweis:

(4) $$\sum_{i=1}^{n} i^4 = ?$$

Voraussetzungen

$$\sum_{i=0}^{n} i^0 = 1n^1 + 1n^0$$

$$\sum_{i=1}^{n} i^1 = \frac{1}{2}n^2 + \frac{1}{2}n^1 + 0\,n^0$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n^1 + 0n^0$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 + 0n^1 + 0n^0$$

$$\sum_{i=0}^{n} i^0 = B_1^0 n^0 + B_0^0 n^0$$

$$\sum_{i=1}^{n} i^1 = B_2^1 n^2 + B_1^1 n^1 + B_0^1 n^0$$

$$\sum_{i=1}^{n} i^2 = B_3^2 n^3 + B_2^2 n^2 + B_2^2 n^1 + B_0^2 n^0$$

$$\sum_{i=1}^{n} i^3 = B_4^3 n^4 + B_3^3 n^3 + B_2^3 n^2 + B_1^3 n^1 + B_0^3 n^0$$

$$\sum_{i=0}^{n} (i+1)^5 = \sum_{i=0}^{n} \binom{5}{5} i^5 + \binom{5}{4} i^4 + \binom{5}{3} i^3 + \binom{5}{2} i^2 + \binom{5}{1} i^1 + \binom{5}{0} i^0$$

$$\sum_{i=1}^{n} i^5 + (n+1)^5 = \sum_{i=1}^{n} \binom{5}{5} i^5 + \sum_{i=1}^{n} \binom{5}{4} i^4 + \sum_{i=1}^{n} \binom{5}{3} i^3 + \sum_{i=1}^{n} \binom{5}{2} i^2 + \sum_{i=1}^{n} \binom{5}{1} i^1 + \sum_{i=0}^{n} \binom{5}{0} i^0$$

$$\sum_{i=1}^{n} i^5 + (n+1)^5 = \binom{5}{5} \sum_{i=1}^{n} i^5 + \binom{5}{4} \sum_{i=1}^{n} i^4 + \binom{5}{3} \sum_{i=1}^{n} i^3 + \binom{5}{2} \sum_{i=1}^{n} i^2 + \binom{5}{1} \sum_{i=1}^{n} i^1 + \binom{5}{0} \sum_{i=0}^{n} i^0$$

$$\binom{5}{4} \sum_{i=1}^{n} i^4 = (n+1)^5 - \left[\binom{5}{3} \sum_{i=1}^{n} i^3 + \binom{5}{2} \sum_{i=1}^{n} i^2 + \binom{5}{1} \sum_{i=1}^{n} i^1 + \binom{5}{0} \sum_{i=0}^{n} i^0 \right]$$

$$\sum_{i=1}^{n} i^4 = \frac{1}{\binom{5}{4}} \left[\binom{5}{5} n^5 + \binom{5}{4} n^4 + \binom{5}{3} n^3 + \binom{5}{2} n^2 + \binom{5}{1} n^1 + \binom{5}{0} n^0 \right]$$

$$- \frac{1}{\binom{5}{4}} \left[\binom{5}{3} \sum_{i=1}^{n} i^3 + \binom{5}{2} \sum_{i=1}^{n} i^2 + \binom{5}{1} \sum_{i=1}^{n} i^1 + \binom{5}{0} \sum_{i=0}^{n} i^0 \right]$$

$$\sum_{i=1}^{n} i^4 = \frac{1}{\binom{5}{4}} \left[\binom{5}{5}n^5 + \binom{5}{4}n^4 + \binom{5}{3}n^3 + \binom{5}{2}n^2 + \binom{5}{1}n^1 + \binom{5}{0}n^0 \right]$$

$$- \frac{1}{\binom{5}{4}} \left[\begin{array}{rrrrr} & & & \binom{5}{0}B_1^0 n^1 + & \binom{5}{0}B_0^0 n^0 \\ & & \binom{5}{1}B_2^1 n^2 + & \binom{5}{1}B_1^1 n^1 + & \binom{5}{1}B_0^1 n^0 \\ & \binom{5}{2}B_3^2 n^3 + & \binom{5}{2}B_2^2 n^2 + & \binom{5}{2}B_1^2 n^1 + & \binom{5}{2}B_0^2 n^0 \\ \binom{5}{3}B_4^3 n^4 + & \binom{5}{3}B_3^3 n^3 + & \binom{5}{3}B_2^3 n^2 + & \binom{5}{3}B_1^3 n^1 + & \binom{5}{3}B_0^3 n^0 \end{array} \right]$$

Hier sieht man jetzt, dass man die Koeffizienten für $\sum_{i=1}^{n} i^4$ wie folgt erhält :

$$B_5^4 = \frac{\binom{5}{5}}{\binom{5}{4}}$$

$$B_4^4 = \frac{\binom{5}{4} - \binom{5}{3}B_4^3}{\binom{5}{4}}$$

$$B_3^4 = \frac{\binom{5}{3} - \binom{5}{3}B_3^3 - \binom{5}{2}B_3^2}{\binom{5}{4}}$$

$$B_2^4 = \frac{\binom{5}{2} - \binom{5}{3}B_2^3 - \binom{5}{2}B_2^2 - \binom{5}{1}B_2^1}{\binom{5}{4}}$$

$$B_1^4 = \frac{\binom{5}{1} - \binom{5}{3}B_1^3 - \binom{5}{2}B_1^2 - \binom{5}{1}B_1^1 - \binom{5}{0}B_1^0}{\binom{5}{4}}$$

$$B_0^4 = \frac{\binom{5}{0} - \binom{5}{3}B_0^3 - \binom{5}{2}B_0^2 - \binom{5}{1}B_0^1 - \binom{5}{0}B_0^0}{\binom{5}{4}}$$

In mehr schematischer Darstellung ergeben sich die Koeffizienten für $\sum_{i=1}^{n} i^4$ wie folgt:

$$
\begin{array}{lcccccc}
 & \binom{5}{5} & \binom{5}{4} & \binom{5}{3} & \binom{5}{2} & \binom{5}{1} & \binom{5}{0} \\
 & & & & & | & | \\
\sum_{i=0}^{n} i^0 & & & & & \binom{5}{0} B_1^0 & \binom{5}{0} B_0^0 \\
 & & & & | & | & | \\
\sum_{i=1}^{n} i^1 & & & & \binom{5}{1} B_2^1 & \binom{5}{1} B_1^1 & \binom{5}{1} B_0^1 \\
 & & & | & | & | & | \\
\sum_{i=1}^{n} i^2 & & & \binom{5}{2} B_3^2 & \binom{5}{2} B_2^2 & \binom{5}{2} B_1^2 & \binom{5}{2} B_0^2 \\
 & & | & | & | & | & | \\
\sum_{i=1}^{n} i^3 & & \binom{5}{3} B_4^3 & \binom{5}{3} B_3^3 & \binom{5}{3} B_2^3 & \binom{5}{3} B_1^3 & \binom{5}{3} B_0^3 \\
 & : & : & : & : & : & : \\
 & \binom{5}{4} & \binom{5}{4} & \binom{5}{4} & \binom{5}{4} & \binom{5}{4} & \binom{5}{4} \\
 & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel \\
\sum_{i=1}^{n} i^4 & B_5^4 & B_4^4 & B_3^4 & B_2^4 & B_1^4 & B_0^4
\end{array}
$$

Konkrete Berechnung der Koeffizienten für $\sum\limits_{i=1}^{n} i^4$:

$$\sum_{i=0}^{n} i^0 = 1n^1 + 1n^0$$

$$\sum_{i=1}^{n} i^1 = \frac{1}{2}n^2 + \frac{1}{2}n^1 + 0\,n^0$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n^1 + 0n^0$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 + 0n^1 + 0n^0$$

	1	5	10	10	5	1
					I	I
$\sum\limits_{i=0}^{n} i^0$					1	1
				I	I	I
$\sum\limits_{i=1}^{n} i^1$				$5\cdot\frac{1}{2}$	$5\cdot\frac{1}{2}$	$5\cdot 0$
			I	I	I	I
$\sum\limits_{i=1}^{n} i^2$			$10\cdot\frac{1}{3}$	$10\cdot\frac{1}{2}$	$10\cdot\frac{1}{6}$	$10\cdot 0$
		I	I	I	I	I
$\sum\limits_{i=1}^{n} i^3$		$10\cdot\frac{1}{4}$	$10\cdot\frac{1}{2}$	$10\cdot\frac{1}{4}$	$10\cdot 0$	$10\cdot 0$
	:	:	:	:	:	:
	5	5	5	5	5	5
	II	II	II	II	II	II
$\sum\limits_{i=1}^{n} i^4$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{3}$	0	$-\frac{1}{30}$	0

$$\boxed{\sum_{i=1}^{n} i^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 + 0n^2 - \frac{1}{30}n^1 + 0\,n^0}$$

Potenzsummen

Zur Begründung der Integralrechnung benötigt man die Potenzsummenformeln:

Für alle $k \in \mathbb{N}$ gilt :

$$\sum_{i=1}^{n} i^k = p_{k+1}(n) = B_{k+1}^k n^{k+1} + B_k^k n^k + \ \ldots \ + B_1^k n^1 + B_0^k n^0$$

mit $B_{k+1}^k = \dfrac{1}{k+1}$.

Beweis: Für $k = 0, 1, 2, 3$

$$\boxed{\sum_{i=1}^{n} i^0 = \sum_{i=1}^{n} 1 = n = \frac{1}{1} n}$$

$$\sum_{i=0}^{n} (i+1)^2 = \sum_{i=0}^{n} i^2+2i+1$$

$$\sum_{i=1}^{n} i^2 + (n+1)^2 = \sum_{i=0}^{n} i^2 + 2\sum_{i=0}^{n} i + \sum_{i=0}^{n} 1$$

$$\sum_{i=1}^{n} i^2 + (n+1)^2 = \sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$$

$$(n+1)^2 = 2\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$$

$$2\sum_{i=1}^{n} i = (n+1)^2 - \sum_{i=0}^{n} 1$$

$$2\sum_{i=1}^{n} i = (n+1)^2 - (n+1)$$

$$2\sum_{i=1}^{n} i = n^2 + 2n + 1 - n - 1$$

$$2\sum_{i=1}^{n} i = n^2 + n$$

$$\boxed{\sum_{i=1}^{n} i = \frac{1}{2}n^2 + \frac{1}{2}n}$$

$$\sum_{i=0}^{n} (i+1)^3 = \sum_{i=0}^{n} i^3+3i^3+3i^2+1$$

$$\sum_{i=1}^{n} i^3 + (n+1)^3 = \sum_{i=0}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i^1 + \sum_{i=0}^{n} 1$$

$$\sum_{i=1}^{n} i^3 + (n+1)^3 = \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i^1 + \sum_{i=0}^{n} 1$$

$$(n+1)^3 = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i^1 + \sum_{i=0}^{n} 1$$

$$3\sum_{i=1}^{n} i^2 = (n+1)^3 - 3\sum_{i=1}^{n} i^1 - \sum_{i=0}^{n} 1$$

$$3\sum_{i=1}^{n} i^2 = (n+1)^3 - 3\frac{n(n+1)}{2} - (n+1)$$

$$6\sum_{i=1}^{n} i^2 = 2(n+1)^3 - 3n(n+1) - 2(n+1)$$

$$6\sum_{i=1}^{n} i^2 = 2n^3+6n^2+6n+2-3n^2-3n-2n-2$$

$$6\sum_{i=1}^{n} i^2 = 2n^3+3n^2+n$$

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n}$$

$$\sum_{i=0}^{n} (i+1)^4 = \sum_{i=0}^{n} i^4+4i^3+6i^2+4i+1$$

$$\sum_{i=1}^{n} i^4 + (n+1)^4 = \sum_{i=0}^{n} i^4 + 4\sum_{i=0}^{n} i^3 + 6\sum_{i=0}^{n} i^2 + 4\sum_{i=0}^{n} i + \sum_{i=0}^{n} 1$$

$$\sum_{i=1}^{n} i^4 + (n+1)^4 = \sum_{i=1}^{n} i^4 + 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$$

$$(n+1)^4 = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$$

$$4\sum_{i=1}^{n} i^3 = (n+1)^4 - 6\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i - \sum_{i=0}^{n} 1$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left((n+1)^4 - 6\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i - \sum_{i=0}^{n} 1\right)$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left((n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - (n+1)\right)$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left(n^4+4n^3+6n^2+4n+1 - 2n^3-3n^2-n - 2n^2-2n - n-1\right)$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left(n^4+n^2+4n+1 - 2n^3-3n^2-n - 2n^2-2n - n-1\right)$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left(n^4+2n^3+n^2\right)$$

$$\boxed{\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n} i^3 = \frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2}$$

Potenzsummen können durch Polynome dargestellt werden

Für alle $k \in \mathbb{N}$ gilt :

$$\sum_{i=1}^{n} i^k = p_{k+1}(n) = B_{k+1}^k n^{k+1} + B_k^k n^k + \ ... \ + B_1^k n^1 + B_0^k n^0$$

mit $B_{k+1}^k = \dfrac{1}{k+1}$.

Beweis durch **Vollständige Induktion** :

(I) Die Behauptung ist richtig für $k = 0$

$$\sum_{i=1}^{n} i^0 = \sum_{i=1}^{n} 1 = n = p_{0+1}(n) = \frac{1}{1} n^{0+1} + 0n^0$$

(II) Die Behauptung sei richtig für irgend ein $k \in \mathbb{N}$ und alle $\bar{k} \leq k$.
Man zeigt, dass sie dann auch für $k + 1$ richtig ist .

$$\sum_{i=0}^{n} (i+1)^{k+2} = \sum_{i=0}^{n} \binom{k+2}{k+2} i^{k+2} + \binom{k+2}{k+1} i^{k+1} + \binom{k+2}{k} i^k + \ ... \ + \binom{k+2}{1} i^1 + \binom{k+2}{0} i^0$$

$$\sum_{i=1}^{n} i^{k+2} + (n+1)^{k+2} = 1 \sum_{i=0}^{n} i^{k+2} + (k+2) \sum_{i=0}^{n} i^{k+1} + \binom{k+2}{k} \sum_{i=0}^{n} i^k + ... + (k+2) \sum_{i=0}^{n} i^1 + n+1$$

$$\sum_{i=1}^{n} i^{k+2} + (n+1)^{k+2} = 1 \sum_{i=1}^{n} i^{k+2} + (k+2) \sum_{i=1}^{n} i^{k+1} + \binom{k+2}{k} \sum_{i=1}^{n} i^k + \ ... \ + (k+2) \sum_{i=1}^{n} i^1 + n+1$$

$$(n+1)^{k+2} = (k+2) \sum_{i=1}^{n} i^{k+1} + \binom{k+2}{k} \sum_{i=1}^{n} i^k + \ ... \ + (k+2) \sum_{i=1}^{n} i^1 + n+1$$

$$(k+2) \sum_{i=1}^{n} i^{k+1} = (n+1)^{k+2} - \binom{k+2}{k} \sum_{i=1}^{n} i^k - \ ... \ - (k+2) \sum_{i=1}^{n} i^1 - (n+1)$$

Nach Induktionsannahme gibt es Polynome $P_{k+1}(n)$, ... , $P_2(n)$, so dass folgt :

$$(k+2)\sum_{i=1}^{n} i^{k+1} = P_{k+2}(n) - P_{k+1}(n) - \dots - P_2(n) - (n+1)$$
$$(k+2)\sum_{i=1}^{n} i^{k+1} = p_{k+2}(n)$$

Nach Zusammenfassung zu einem Polynom $p_{k+2}(n)$ mit dem ersten Glied $1n^{k+2}$ und entsprechender Division ergibt sich :

$$(k+2)\sum_{i=1}^{n} i^{k+1} = 1n^{k+2} + \dots$$

$$\sum_{i=1}^{n} i^{k+1} = \frac{1}{(k+2)} n^{k+2} + \dots$$

q.e.d.

Bestimmung der Polynomkoeffizienten über ein lineares Gleichungssystem

$$\sum_{i=1}^{n} i^3 = p_4(n) = a_4 n^4 + a_3 n^3 = a_2 n^2 + a_1 n + a_0$$

$$\begin{aligned}
n=1: \quad & 1 &= a_4 &+ a_3 &+ a_2 &+ a_1 &+ a_0 \\
n=2: \quad & 9 &= 16a_4 &+ 8a_3 &+ 4a_2 &+ 2_1 &+ a_0 \\
n=3: \quad & 36 &= 81a_4 &+ 27a_3 &+ 9a_2 &+ 3a_1 &+ a_0 \\
n=4: \quad & 100 &= 256a_4 &+ 64a_3 &+ 16a_2 &+ 4a_1 &+ a_0 \\
n=5: \quad & 225 &= 625a_4 &+ 125a_3 &+ 25a_2 &+ 5a_1 &+ a_0
\end{aligned}$$

$$\begin{aligned}
8 &= 15a_4 + 7a_3 + 3a_2 + a_1 \\
27 &= 65a_4 + 19a_3 + 5a_2 + a_1 \\
64 &= 175a_4 + 37a_3 + 7a_2 + a_1 \\
125 &= 369a_4 + 61a_3 + 7a_2 + a_1
\end{aligned}$$

$$\begin{aligned}
19 &= 50a + 12a_3 + 2a_2 \\
37 &= 110a + 18a_3 + 2a_2 \\
61 &= 194a + 24a_3 + 2a_2
\end{aligned}$$

$$\begin{aligned}
18 &= 60a_4 + 6a_3 \\
24 &= 84a_4 + 6a_3 \\
6 &= 24a_4
\end{aligned}$$

$$\boxed{a_4 = \frac{1}{4}} \quad \boxed{a_3 = \frac{1}{2}}$$

$$\begin{aligned}
19 &= 50\frac{1}{4} + 6 + 2a_2 \\
19 &= \frac{25}{2} + 6 + 2a_2 \\
38 &= 25 + 12 + 4a_2
\end{aligned}$$

$$\boxed{a_2 = \frac{1}{4}} \quad \boxed{a_1 = 0} \quad \boxed{a_0 = 0}$$

$$\boxed{\sum_{i=1}^{n} i^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2}$$