Flächenschwerpunkte

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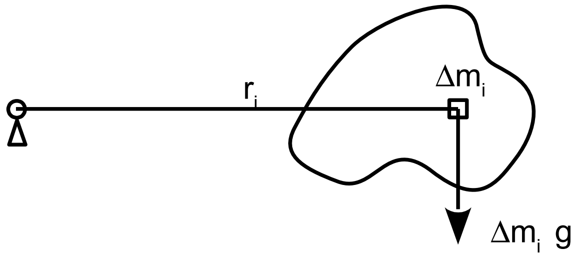
Das Konzept des Schwerpunktes

Gegeben sei ein Körper der Masse m, welche in kleine Teile der Masse Δm_i unterteilt ist.
Das Drehmoment M_i der Masse Δm_i ist gegeben durch

$$M_i = r_i \cdot \Delta m_i \cdot g \ ,$$

wobei r_i der Hebelarm und g die Fallbeschleunigung sind .



Das Drehmoment des Körpers ergibt sich näherungsweise durch

$$M = \sum_i r_i \cdot \Delta m_i \cdot g \ .$$

Bei einem homogenen Körper der Dichte ρ und $\Delta m_i = \rho \cdot \Delta V_i$ ergibt sich

$$M = \sum_i r_i \cdot \rho \Delta V_i \cdot g \ .$$

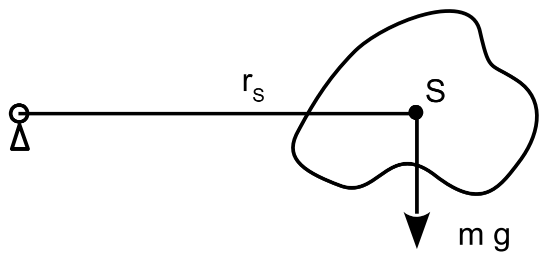
Die Grenzwertbetrachtung $\lim\limits_{i \to \infty} \sum\limits_i r_i \cdot \rho \cdot \Delta V_i \cdot g$ liefert $M = \int\limits_V r \cdot \rho \cdot g \ dV$.

Denkt man sich nun die gesamte Masse des Körpers im Schwerpunkt S mit dem zugehörigen Hebelarm r_S vereinigt, erhält man

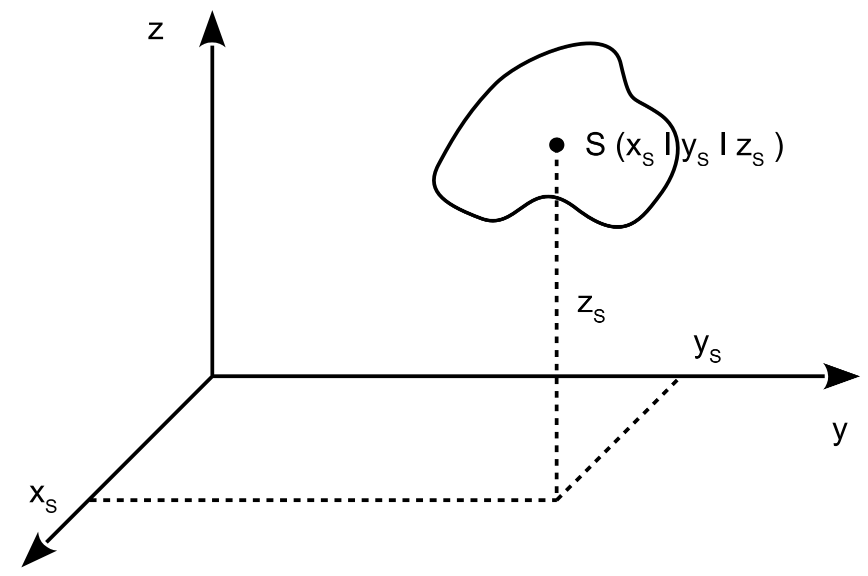
$$M = \int_V r \cdot \rho \cdot g \ dV \ =: \ r_S \cdot \rho \cdot V \cdot g$$

$$\int_V r \ dV \ =: \ r_S \cdot V$$

$$\boxed{\frac{1}{V} \int_V r \ dV \ =: \ r_S}$$



Entsprechend definiert man für einen Körper im Koordinatensystem

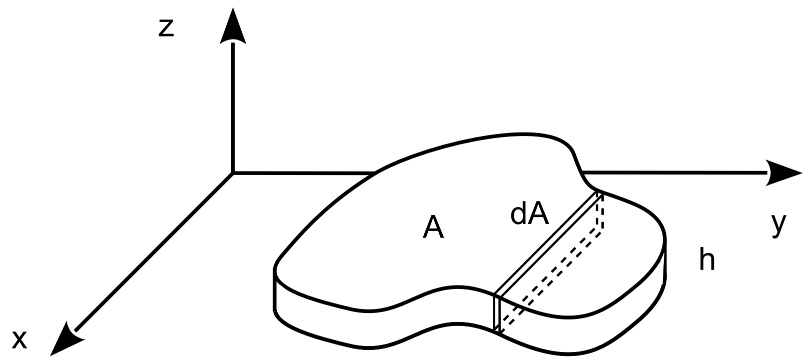


$$x_S := \frac{1}{V} \int_V x \; dV$$

$$y_S := \frac{1}{V} \int_V y \; dV$$

$$z_S := \frac{1}{V} \int_V z \; dV$$

Schwerpunkt einer Scheibe der Grundfläche A **Dicke** h



$$x_S = \frac{1}{V} \int_V x \; dV$$

$$x_S = \frac{1}{A h} \int_A x h \; dA$$

$$x_S = \frac{1}{A} \int_A x \; dA \qquad y_S = \frac{1}{A} \int_A y \; dA$$

$$z_S = \frac{1}{V} \int_V z \; dV$$

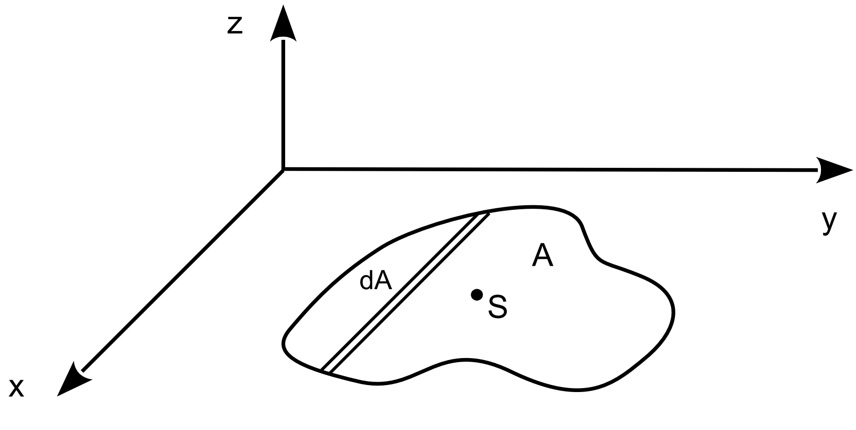
$$z_S = \frac{1}{Ah} \int_0^h z A \; dz$$

$$z_S = \frac{1}{h} \int_0^h z \; dz$$

$$z_S = \left| \frac{1}{h} \frac{z^2}{2} \right|_0^h$$

$$z_S = \frac{h}{2}$$

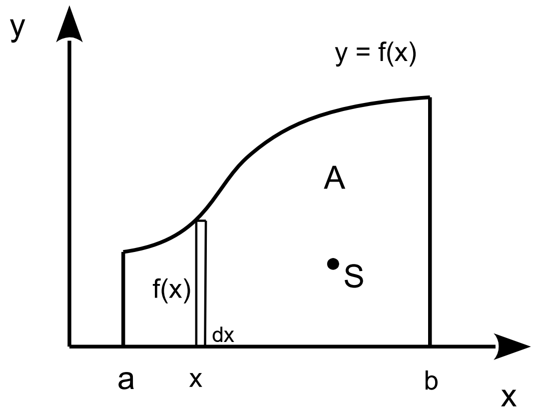
Schwerpunkt einer homogenen Fläche A

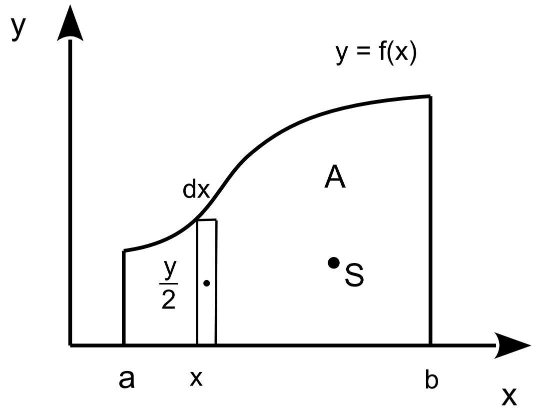


$$\boxed{x_S = \frac{1}{A} \int_A x \; dA}$$

$$\boxed{y_S = \frac{1}{A} \int_A y \; dA}$$

Schwerpunkt einer Fläche unter der Kurve $y = f(x)$ **von** a **bis** b





$$x_S = \frac{1}{A} \int_A x \; dA$$

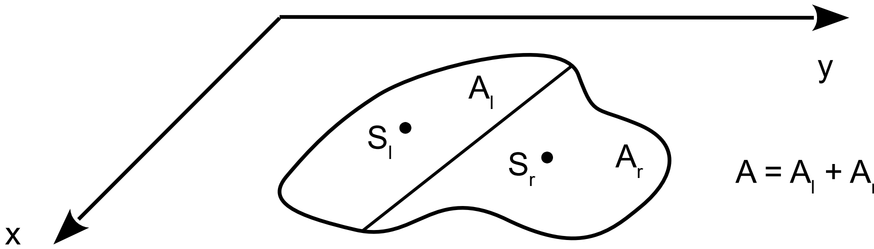
$$\boxed{x_S = \frac{1}{A} \int_a^b x \cdot y \; dx}$$

$$y_S = \frac{1}{A} \int_a^b \frac{y}{2} \; dA$$

$$y_S = \frac{1}{A} \int_a^b \frac{y}{2} \; y \; dx$$

$$\boxed{y_S = \frac{1}{A} \int_a^b \frac{y^2}{2} \; dx}$$

Der Schwerpunkt als Summe gewichteter Schwerpunkte (Summensatz)



$A = A_l + A_r$

$$x_S = \frac{1}{A} \int_A x \; dA$$

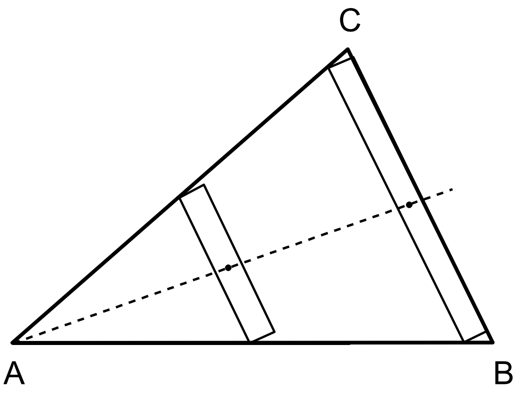
$$x_S = \frac{1}{A} \int_{A_l} x \; dA + \frac{1}{A} \int_{A_r} x \; dA$$

$$x_S = \frac{A_l}{A} \frac{1}{A_l} \int_{A_l} x \; dA + \frac{A_r}{A} \frac{1}{A_r} \int_{A_r} x \; dA$$

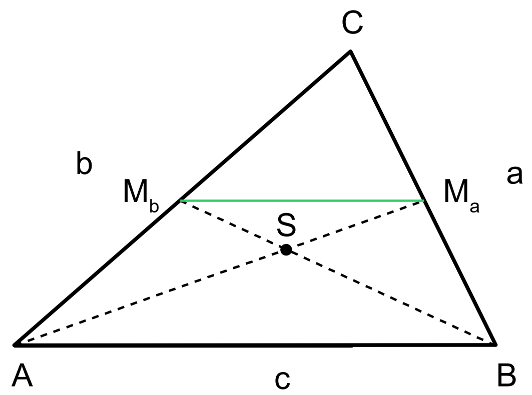
$$x_S = \frac{A_l}{A} x_l + \frac{A_r}{A} x_r$$

$$\boxed{x_S = \frac{A_l}{A_l + A_r} x_l + \frac{A_r}{A_l + A_r} x_r} \quad \text{analog} \quad \boxed{y_S = \frac{A_l}{A_l + A_r} y_l + \frac{A_r}{A_l + A_r} y_r}$$

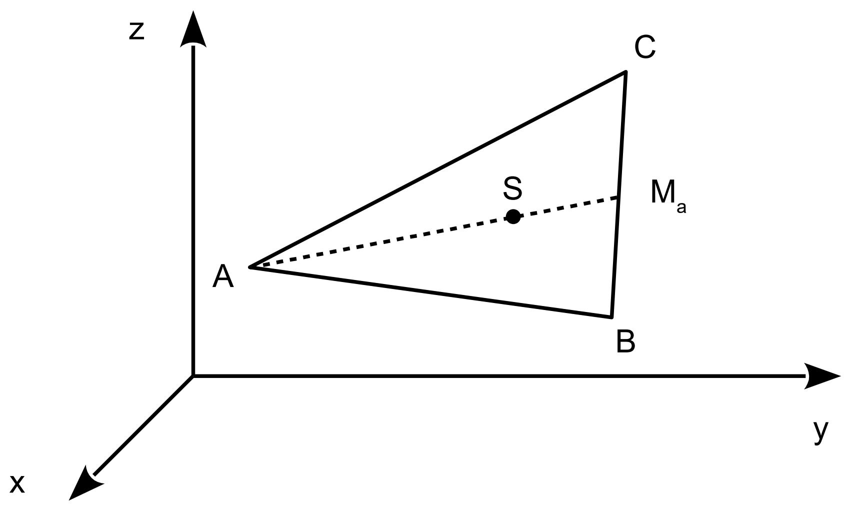
Schwerpunkt des Dreiecks (1. Variante)



Denkt man sich das Dreieck wie oben in in lauter Rechtecke zerlegt, so liegen die Schwerpunkte der Rechtecke auf der Seitenhalbierenden, und damit liegt auch der Schwerpunkt des Dreiecks auf der Seitenhalbierenden von $\overline{BC}$.
Entsprechend liegt der Schwerpunkt auf der Seitenhalbierenden von $\overline{AC}$.



$$\frac{AS}{SM_a} = \frac{c}{M_aM_b} = \frac{a}{\frac{a}{2}} = 2 \quad \Rightarrow \quad AS = 2SM_a \quad \Rightarrow \quad AS = \frac{2}{3}AM_a$$



$$M_a\left(\frac{x_B+x_C}{2} \middle| \frac{y_B+y_C}{2} \middle| \frac{z_B+z_C}{2} \right)$$

$$\begin{pmatrix} x_S \\ y_S \\ z_s \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix} + \frac{2}{3}\begin{pmatrix} \frac{x_B+x_C}{2} - x_A \\ \frac{y_B+y_C}{2} - y_A \\ \frac{z_B+z_C}{2} - z_A \end{pmatrix}$$

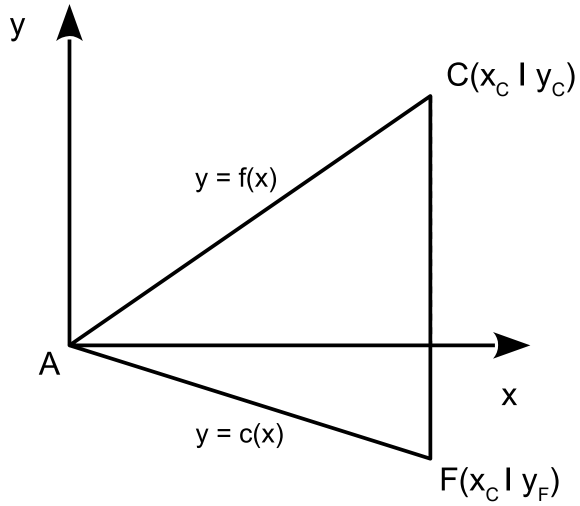
$$\begin{pmatrix} x_S \\ y_S \\ z_s \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix} + \begin{pmatrix} \frac{x_B+x_C}{3} - \frac{2}{3}x_A \\ \frac{y_B+y_C}{3} - \frac{2}{3}y_A \\ \frac{z_B+z_C}{3} - \frac{2}{3}z_A \end{pmatrix}$$

$$\begin{pmatrix} x_S \\ y_S \\ z_s \end{pmatrix} = \begin{pmatrix} \frac{x_A}{3} + \frac{x_B+x_C}{3} \\ \frac{y_A}{3} + \frac{y_B+y_C}{3} \\ \frac{z_A}{3} + \frac{z_B+z_C}{3} \end{pmatrix}$$

$$\begin{pmatrix} x_S \\ y_S \\ z_s \end{pmatrix} = \begin{pmatrix} \frac{x_A+x_B+x_C}{3} \\ \frac{y_A+y_B+y_C}{3} \\ \frac{z_A+z_B+z_C}{3} \end{pmatrix}$$

$$\boxed{S\left(\frac{x_A+x_B+x_C}{3} \middle| \frac{y_A+y_B+y_C}{3} \middle| \frac{z_A+z_B+z_C}{3} \right)}$$

Schwerpunkt des Dreiecks (2. Variante)



$$f: \quad y = \frac{y_C}{x_C}x$$

$$c: \quad y = \frac{y_F}{x_C}x$$

$$A_{AFC} = \frac{(y_C - y_F)\,x_c}{2}$$

$$x_S = \frac{1}{A_{AFC}} \int_0^{x_C} x\left(f(x) - c(x)\right)\,dx$$

$$x_S = \frac{1}{A_{AFC}} \int_0^{x_C} x\left(\frac{y_C}{x_C}x - \frac{y_F}{x_C}x\right)\,dx$$

$$x_S = \frac{1}{A_{AFC}} \int_0^{x_C} \frac{y_C - y_F}{x_C}\,x^2\,dx$$

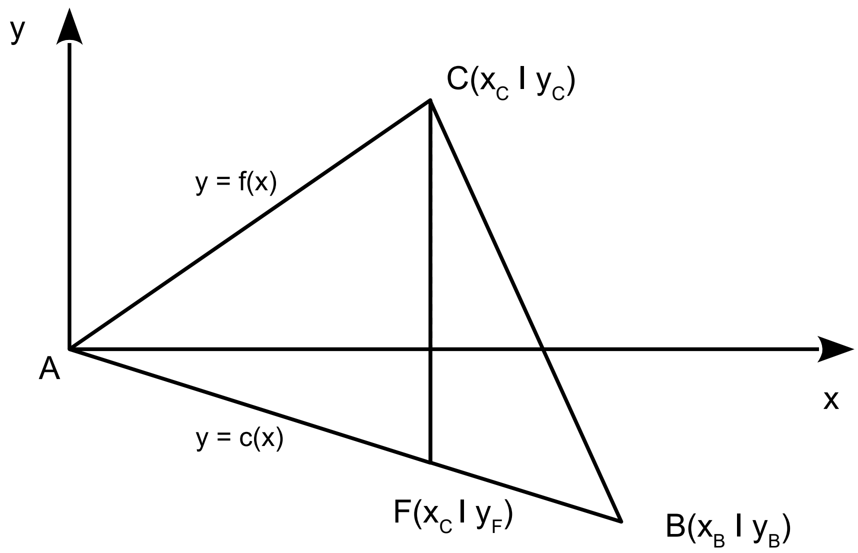
$$x_S = \frac{2}{(y_C - y_F)\,x_c} \int_0^{x_C} \frac{y_C - y_F}{x_C}\,x^2\,dx$$

$$x_S = \frac{2}{x_c^2} \int_0^{x_C} x^2\,dx$$

$$x_S = \frac{2}{x_c^2} \left[\frac{1}{3}x^3\right]_0^{x_C}$$

$$x_S = \frac{2}{x_c^2}\,\frac{1}{3}x_C^3$$

$$x_S = \frac{2}{3}\,x_C$$



$$A_{AFC} = \frac{(y_C - y_F)\,x_c}{2}$$

$$A_{FBC} = \frac{(y_C - y_F)(x_B - x_C)}{2}$$

$$A_{ABC} = A_{AFC} + A_{FBC}$$
$$A_{ABC} = \frac{(y_C - y_F)\,x_c}{2} + \frac{(y_C - y_F)(x_B - x_C)}{2}$$
$$A_{ABC} = \frac{(y_C - y_F)\,x_B}{2}$$

$$\frac{A_{AFC}}{A_{ABC}} = \frac{\frac{(y_C - y_F)\,x_C}{2}}{\frac{(y_C - y_F)\,x_B}{2}} = \frac{x_C}{x_B}$$

$$\frac{A_{FBC}}{A_{ABC}} = \frac{\frac{(y_C - y_F)(x_B - x_C)}{2}}{\frac{(y_C - y_F)\,x_B}{2}} = \frac{x_B - x_C}{x_B}$$

$$x_l = \frac{2}{3}\;x_C$$

$$x_r = x_B - \frac{2}{3}\;(x_B - x_C)$$

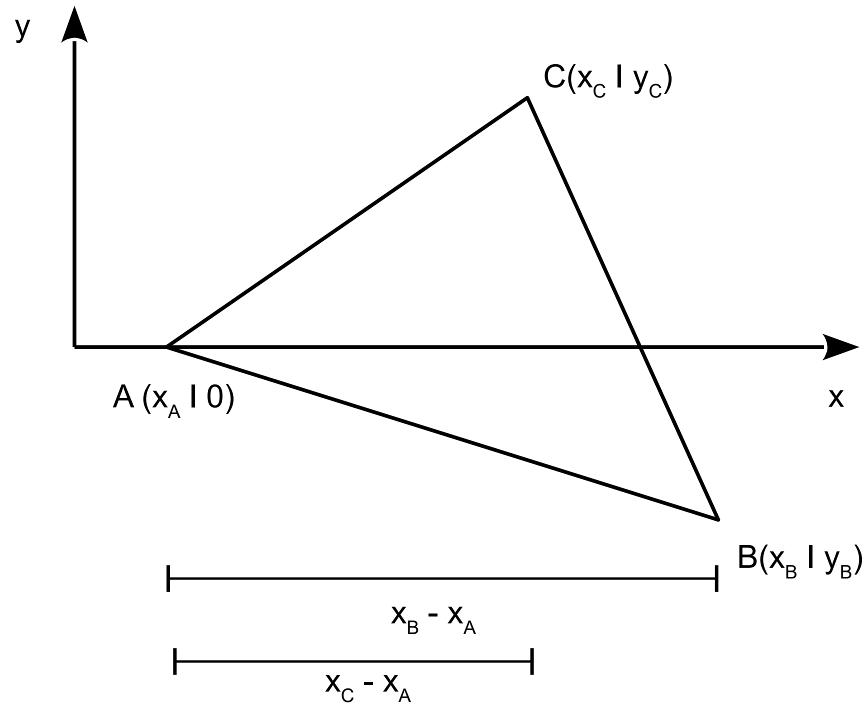
$$x_r = \frac{3x_B - 2\,x_B + 2\,x_C}{3}$$

$$x_r = \frac{x_B + 2\,x_C}{3}$$

Summensatz :

$$x_S = \frac{A_{AFC}}{A_{ABC}}\;x_l + \frac{A_{FBC}}{A_{ABC}}\;x_r$$
$$x_S = \frac{x_C}{x_B}\;\frac{2}{3}\;x_C + \frac{x_B - x_C}{x_B}\;\frac{x_B + 2\,x_C}{3}$$
$$x_S = \frac{x_C}{x_B}\;\frac{2}{3}\;x_C + \frac{x_B - x_C}{x_B}\;\frac{x_B + 2\,x_C}{3}$$
$$x_S = \frac{2\,x_C^2 + x_B^2 + x_B\,x_C - 2c_C^2}{3\,x_B}$$
$$x_S = \frac{x_B^2 + x_B\,x_C}{3\,x_B}$$
$$x_S = \frac{x_B(x_B + x_C)}{3\,x_B}$$
$$x_S = \frac{x_B + x_C}{3}$$

Waagrechte Verschiebung des Dreiecks :

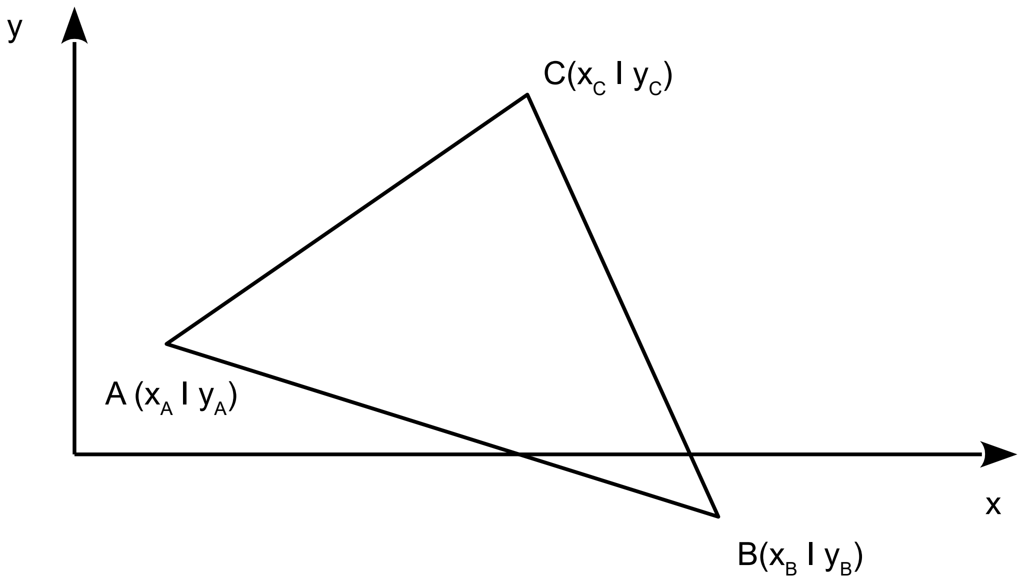


$$x_S = x_A + \frac{x_B - x_A + x_C - x_A}{3}$$

$$x_S = \frac{3x_A + x_B - x_A + x_C - x_A}{3}$$

$$x_S = \frac{x_A + x_B + x_C}{3}$$

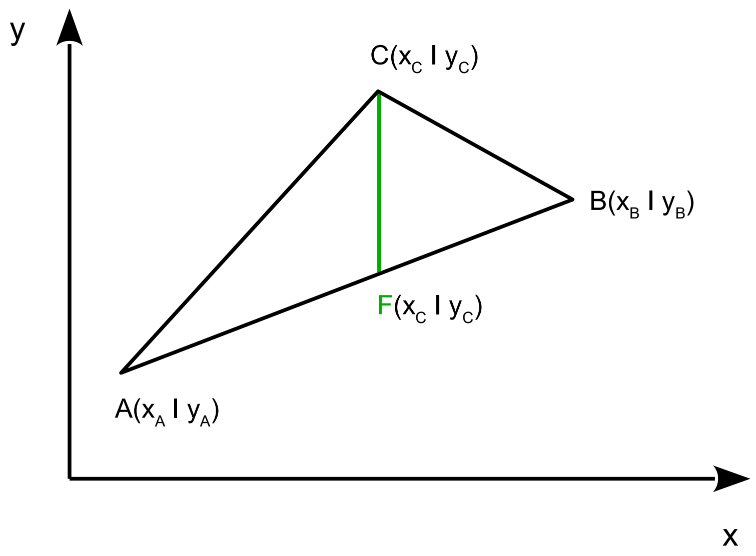
Senkrechte Verschiebung des Dreiecks :



$$\boxed{x_S = \frac{x_A + x_B + x_C}{3}}$$

Wegen Symmetrie : $\boxed{y_S = \frac{y_A + y_B + y_C}{3}}$

Schwerpunkt des Dreiecks (3. Variante)



$$A_{AFC} = \frac{1}{2}(y_C - y_F)(x_C - x_A) \qquad A_{BFC} = \frac{1}{2}(y_C - y_F)(x_B - x_C)$$

$$A_{ABC} = \frac{1}{2}(y_C - y_F)(x_C - x_A) + \frac{1}{2}(y_C - y_F)(x_B - x_C)$$

$$A_{ABC} = \frac{1}{2}(y_C - y_F)(x_C - x_A + x_B - x_C)$$

$$A_{ABC} = \frac{1}{2}(y_C - y_F)(x_B - x_A)$$

$$\frac{A_{AFC}}{A_{ABC}} = \frac{x_C - x_A}{x_B - x_A} \qquad \frac{A_{BFC}}{A_{ABC}} = \frac{x_B - x_C}{x_B - x_A}$$

Berechnung der x- Koordinate des Schwerpunkts des Dreiecks ΔAFC **:**

$$\frac{1}{A_{AFC}} \int_{x_A}^{x_C} \left(\frac{y_C - y_A}{x_C - x_A}(x - x_A) + y_A - \frac{y_F - y_A}{x_C - x_A}(x - x_A) - y_A \right) x \, dx$$

$$\frac{1}{A_{AFC}} \int_{x_A}^{x_C} \left(\frac{y_C - y_A}{x_C - x_A}(x - x_A) - \frac{y_F - y_A}{x_C - x_A}(x - x_A) \right) x \, dx$$

$$\frac{1}{A_{AFC}} \int_{x_A}^{x_C} \left(\left(\frac{y_C - y_A}{x_C - x_A} - \frac{y_F - y_A}{x_C - x_A} \right)(x - x_A) \right) x \, dx$$

$$\frac{1}{A_{AFC}} \int_{x_A}^{x_C} \frac{y_C - y_A - y_F + y_A}{x_C - x_A}(x - x_A) x \, dx$$

$$\frac{1}{A_{AFC}} \int_{x_A}^{x_C} \frac{y_C - y_F}{x_C - x_A}(x - x_A) x \, dx$$

$$\frac{1}{A_{AFC}} \int_{x_A}^{x_C} \frac{y_C-y_F}{x_C-x_A}\left(x^2-x_A x\right)\ dx$$

$$\frac{1}{A_{AFC}} \frac{y_C-y_F}{x_C-x_A} \int_{x_A}^{x_C} x^2-x_A x\ dx$$

$$\frac{2}{\left(y_C-y_F\right)\left(x_C-x_A\right)} \frac{y_C-y_F}{x_C-x_A} \int_{x_A}^{x_C} x^2-x_A x\ dx$$

$$\frac{2}{\left(x_C-x_A\right)^2} \int_{x_A}^{x_C} x^2-x_A x\ dx$$

$$\frac{2}{\left(x_C-x_A\right)^2} \left(\int_{x_A}^{x_C} x^2\ dx\ -\ \int_{x_A}^{x_C} x_A x\ dx\right)$$

$$\frac{2}{\left(x_C-x_A\right)^2} \left(\ \frac{1}{3}\ \left[x^3\right]_{x_A}^{x_C}\ -\ \frac{1}{2}\ x_A\ \left[x^2\right]_{x_A}^{x_C}\ \right)$$

$$\frac{2}{\left(x_C-x_A\right)^2} \left(\ \frac{1}{3}\ \left(x_C^3\ -\ x_A^3\right)\ -\ \frac{1}{2}\ x_A\ \left(x_C^2\ -\ x_A^2\right)\ \right)$$

$$\frac{2}{\left(x_C-x_A\right)^2} \left(\ \frac{1}{3}\ \left(x_C^3\ -\ x_A^3\right)\ -\ \frac{1}{2}\ x_A\ \left(x_C^2\ -\ x_A^2\right)\ \right)$$

$$\frac{2}{\left(x_C-x_A\right)^2} \left(\ \frac{1}{3}\ \left(x_C-x_A\right)\left(x_C^2+x_c x_A+x_A^2\right)\ -\ \frac{1}{2}\ x_A\ \left(x_C-x_A\right)\left(x_C+x_A\right)\ \right)$$

$$\frac{2}{\left(x_C-x_A\right)} \left(\ \frac{1}{3}\ \left(x_C^2+x_c x_A+x_A^2\right)\ -\ \frac{1}{2}\ x_A\ \left(x_C+x_A\right)\ \right)$$

$$\frac{2}{\left(x_C-x_A\right)} \frac{1}{6} \left(\ 2\ \left(x_C^2+x_c x_A+x_A^2\right)\ -\ 3\ x_A\ \left(x_C+x_A\right)\ \right)$$

$$\frac{2}{\left(x_C-x_A\right)} \frac{1}{6} \left(\ 2x_C^2\ +\ 2x_c x_A\ +\ 2x_A^2\ -\ 3x_A x_C\ -\ 3x_A^2\ \right)$$

$$\frac{2}{\left(x_C-x_A\right)} \frac{1}{6} \left(\ 2x_C^2\ -\ x_c x_A\ -\ x_A^2\ \right)$$

$$\frac{2}{\left(x_C-x_A\right)} \frac{1}{6} \left(\ 2x_C^2\ -\ 2x_c x_A\ +\ x_C x_A\ -\ x_A^2\ \right)$$

$$\frac{2}{\left(x_C-x_A\right)} \frac{1}{6} \left(\ 2x_C\left(x_C-x_A\right)\ +\ x_A\left(x_C-x_A\right)\ \right)$$

$$\frac{2}{1} \frac{1}{6} \left(\ 2x_C\ +\ x_A\ \right)$$

$$\frac{2x_C\ +\ x_A}{3}$$

Die x-Koordinate des Schwerpunkts des Dreiecks ΔAFC **ist also**

$$\frac{2x_C\ +\ x_A}{3}\quad .$$

Analog ist die x-Koordinate des Dreiecks ΔBFC **gleich**

$$\frac{2x_C\ +\ x_B}{3}\quad .$$

Nach dem **Summensatz** ist die x- Koordinate des Schwerpunkts von Δ ABC :

$$x_S = \frac{x_C - x_A}{x_B - x_A}\,\frac{2x_C + x_A}{3} + \frac{x_B - x_C}{x_B - x_A}\,\frac{2x_C + x_B}{3}$$

$$x_S = \frac{\left(x_C - x_A\right)\left(2x_C + x_A\right) + \left(x_B - x_C\right)\left(2x_C + x_B\right)}{3\left(x_B - x_A\right)}$$

$$x_S = \frac{2x_c^2 + x_C x_A - 2x_C x_A - x_A^2 + 2x_B x_C + x_B^2 - 2x_C^2 - x_B x_C}{3\left(x_B - x_A\right)}$$

$$x_S = \frac{- x_C x_A - x_A^2 + x_B x_C + x_B^2}{3\left(x_B - x_A\right)}$$

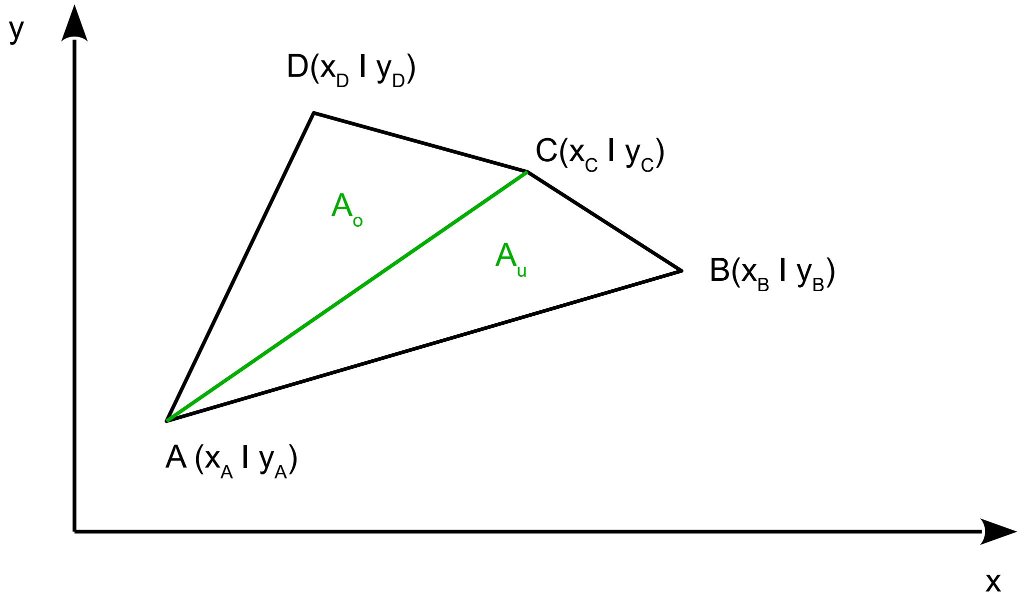
$$x_S = \frac{- x_C x_A + x_B x_C - x_A^2 + x_B^2}{3\left(x_B - x_A\right)}$$

$$x_S = \frac{x_C\left(x_B - x_A\right) + \left(x_B + x_A\right)\left(x_B - x_A\right)}{3\left(x_B - x_A\right)}$$

$$x_S = \frac{x_C + \left(x_B + x_A\right)}{3}$$

$$\boxed{x_S = \frac{x_A + x_B + x_C}{3}} \qquad \text{Wegen Symmetrie :} \qquad \boxed{y_S = \frac{y_A + y_B + y_C}{3}}$$

Schwerpunkt des Allgemeinen Vierecks



Gaußsche Trapezformeln für den den Flächeninhalt von Polygonen

$$A_u = \frac{1}{2}\left(\left(y_A+y_B\right)\left(x_A-x_B\right) + \left(y_B+y_C\right)\left(x_B-x_C\right) + \left(y_C+y_A\right)\left(x_C-x_A\right)\right)$$

$$A_o = \frac{1}{2}\left(\left(y_A+y_C\right)\left(x_A-x_C\right) + \left(y_C+y_D\right)\left(x_C-x_D\right) + \left(y_D+y_A\right)\left(x_D-x_A\right)\right)$$

$$A_u+A_0 = \frac{1}{2}\left(\left(y_A+y_B\right)\left(x_A-x_B\right) + \left(y_B+y_C\right)\left(x_B-x_C\right) + \left(y_C+y_D\right)\left(x_C-x_D\right) + \left(y_D+y_A\right)\left(x_D-x_A\right)\right)$$

$$S_u\left(x_u|y_u\right) = S_u\left(\frac{x_A+x_B+x_C}{3} \;\middle|\; \frac{y_A+y_B+y_C}{3}\right)$$

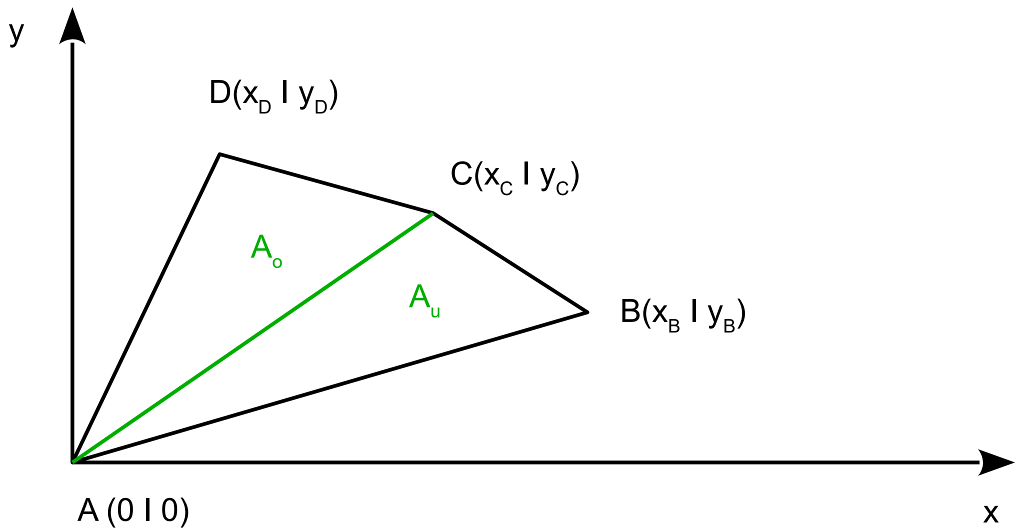
$$S_o\left(x_o|y_o\right) = S_o\left(\frac{x_A+x_C+x_D}{3} \;\middle|\; \frac{y_A+y_C+y_D}{3}\right)$$

Summensatz:

$$\boxed{x_S = \frac{A_u}{A_u+A_o}\,x_u + \frac{A_o}{A_u+A_o}\,x_o} \quad , \quad \boxed{y_S = \frac{A_u}{A_u+A_o}\,y_u + \frac{A_o}{A_u+A_o}\,y_o}$$

Spezialfälle

(1) Allgemeines Viereck in spezieller Lage, $A(0 \mid 0)$



$$A_u = \frac{1}{2}\left((y_A+y_B)(x_A-x_B) + (y_B+y_C)(x_B-x_C) + (y_C+y_A)(x_C-x_A)\right)$$
$$A_u = \frac{1}{2}\left((0+y_B)(0-x_B) + (y_B+y_C)(x_B-x_C) + (y_C+0)(x_C-0)\right)$$
$$A_u = \frac{1}{2}\left(-\ y_B x_B + (y_B+y_C)(x_B-x_C) + y_C x_C\right)$$
$$A_u = \frac{1}{2}\left(-\ y_B x_B + y_B x_B - y_B x_C + y_C x_B - y_c x_C + y_C x_C\right)$$
$$A_u = \frac{1}{2}\left(-\ y_B x_C + y_C x_B\right)$$

$$A_o = \frac{1}{2}\left((y_A+y_C)(x_A-x_C) + (y_C+y_D)(x_C-x_D) + (y_D+y_A)(x_D-x_A)\right)$$
$$A_o = \frac{1}{2}\left((0+y_C)(0-x_C) + (y_C+y_D)(x_C-x_D) + (y_D+0)(x_D-0)\right)$$
$$A_o = \frac{1}{2}\left(-\ y_C x_C + (y_C+y_D)(x_C-x_D) + y_D x_D\right)$$
$$A_o = \frac{1}{2}\left(-\ y_C x_C + y_C x_C - y_C x_D + y_D x_C - y_D x_D + y_D x_D\right)$$
$$A_o = \frac{1}{2}\left(-\ y_C x_D + y_D x_C\right)$$

$$A_u + A_0 = \frac{1}{2}\left(-\ y_B x_C + y_C x_B - y_C x_D + y_D x_C\right)$$

$$S_u(x_u|y_u) = S_u\left(\frac{x_B+x_C}{3} \;\middle|\; \frac{y_B+y_C}{3}\right)$$
$$S_o(x_o|y_o) = S_o\left(\frac{x_C+x_D}{3} \;\middle|\; \frac{y_C+y_D}{3}\right)$$

Summensatz:

$$x_S = \frac{A_u}{A_u + A_o}\, x_u + \frac{A_o}{A_u + A_o}\, x_o$$

$$x_S = \frac{-y_B x_C + y_C x_B}{-y_B x_C + y_C x_B - y_C x_D + y_D x_C}\,\frac{x_B + x_C}{3} + \frac{-y_C x_D + y_D x_C}{-y_B x_C + y_C x_B - y_C x_D + y_D x_C}\,\frac{x_C + x_D}{3}$$

$$\boxed{x_S = \frac{\left(-y_B x_C + y_C x_B\right)\left(x_B + x_C\right) + \left(-y_C x_D + y_D x_C\right)\left(x_C + x_D\right)}{3\left(-y_B x_C + y_C x_B - y_C x_D + y_D x_C\right)}}$$

Analog

$$\boxed{x_S = \frac{\left(-y_B x_C + y_C x_B\right)\left(y_B + y_C\right) + \left(-y_C x_D + y_D x_C\right)\left(y_C + y_D\right)}{3\left(-y_B x_C + y_C x_B - y_C x_D + y_D x_C\right)}}$$

(2) Für $D = C$ ergibt sich das Dreieck ΔABC und $A_o = 0$

$$x_S = \frac{A_u}{A_u + A_o}\, x_u + \frac{A_o}{A_u + A_o}\, x_o$$

$$x_S = \frac{A_u}{A_u + 0}\, x_u + \frac{0}{A_u + A_o}\, x_o$$

$$x_S = x_u$$

$$\boxed{x_S = \frac{x_A + x_B + x_C}{3}} \text{ , analog } \boxed{y_S = \frac{y_A + y_B + y_C}{3}}$$

(3) Die Punkte B , C sind gleich weit von der Diagonale $\overline{AC}$ entfernt.

$$A_o = A_u$$

$$x_S = \frac{A_u}{A_u + A_o}\, x_u + \frac{A_o}{A_u + A_o}\, x_o$$

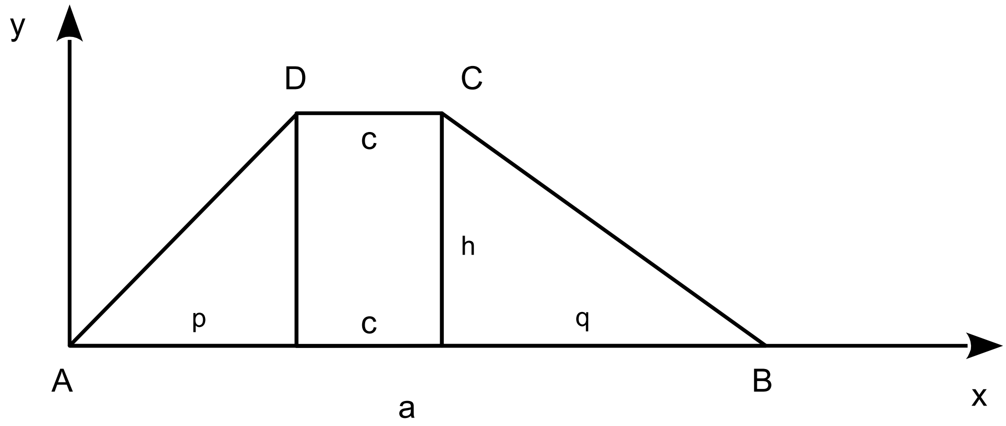
$$x_S = \frac{1}{2}\, x_u + \frac{1}{2}\, x_o$$

$$x_S = \frac{1}{2}\left(x_u + x_o\right)$$

$$x_S = \frac{1}{2}\left(\frac{x_A + x_B + x_C}{3} + \frac{x_A + x_C + x_D}{3}\right)$$

$$\boxed{x_S = \frac{1}{2}\left(\frac{2x_A + x_B + 2x_C + x_D}{3}\right)} \text{ , analog } \boxed{y_S = \frac{1}{2}\left(\frac{2y_A + y_B + 2y_C + y_D}{3}\right)}$$

(4) Trapez $\quad A(0|0)\ ,\quad B(a|0)\ ,\quad C(p+c|h)\ ,\quad D(p|h)\ ,\quad p+c+q = a$



$$A_u = \frac{ah}{2} \quad , \quad A_o = \frac{ch}{2} \quad , \quad A_u + A_o = \frac{(a+c)h}{2}$$

$$\frac{A_u}{A_u+A_o} = \frac{a}{a+c} \quad , \quad \frac{A_o}{A_u+A_o} = \frac{c}{a+c}$$

$$S_u(x_u|y_u) = S_u\left(\frac{a+p+c}{3} \;\middle|\; \frac{h}{3}\right)$$

$$S_o(x_o|y_o) = S_u\left(\frac{2p+c}{3} \;\middle|\; \frac{2h}{3}\right)$$

$$x_S = \frac{A_u}{A_u+A_o}\, x_u + \frac{A_o}{A_u+A_o}\, x_o$$

$$x_S = \frac{a}{a+c}\,\frac{a+p+c}{3} + \frac{c}{a+c}\,\frac{2p+c}{3}$$

$$x_S = \frac{a^2+ap+ac+2pc+c^2}{3(a+c)}$$

$$x_S = \frac{a^2-c^2+ap+ac+2pc+2c^2}{3(a+c)}$$

$$x_S = \frac{a^2-c^2+a(p+c)+2c(p+c)}{3(a+c)}$$

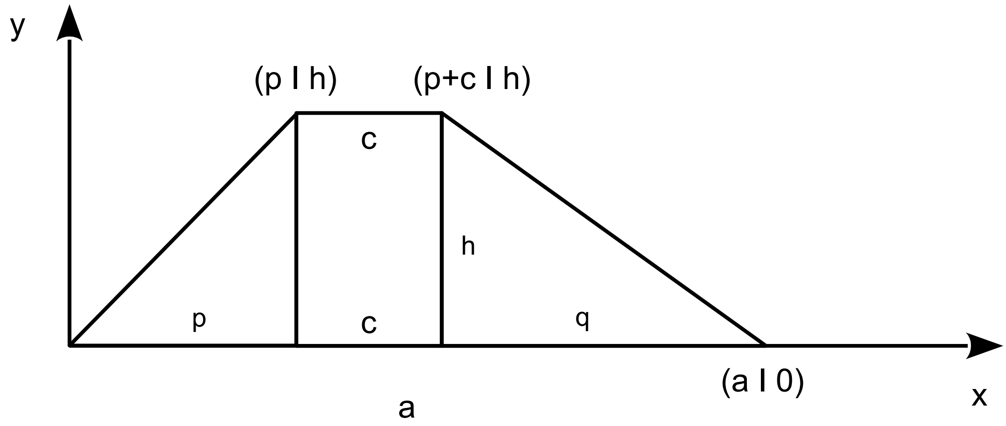
$$\boxed{x_S = \frac{a^2 - c^2 + (a+2c)(p+c)}{3(a+c)}}$$

$$y_S = \frac{A_u}{A_u+A_o}\, y_u + \frac{A_o}{A_u+A_o}\, y_o$$

$$y_S = \frac{a}{a+c}\,\frac{h}{3} + \frac{c}{a+c}\,\frac{2h}{3}$$

$$\boxed{y_S = \frac{a+2c}{a+c}\,\frac{h}{3}}$$

Schwerpunkt des Trapezes



$$A_l = \frac{ph}{2} \qquad A_m = ch \qquad A_r = \frac{qh}{2}$$

$$A = \frac{a+c}{2}\,h$$

$$\frac{A_l}{A} = \frac{\frac{ph}{2}}{\frac{a+c}{2}\,h} = \frac{p}{a+c} \qquad \frac{A_m}{A} = \frac{ch}{\frac{a+c}{2}\,h} = \frac{2c}{a+c} \qquad \frac{A_l}{A} = \frac{\frac{qh}{2}}{\frac{a+c}{2}\,h} = \frac{q}{a+c}$$

$$x_l = \frac{2}{3}\,p \qquad x_m = p+\frac{c}{2} = \frac{2p+c}{2} \qquad x_r = p+c+\frac{1}{3}q = \frac{3p+3c+q}{3}$$

$$y_l = \frac{h}{3} \qquad y_m = \frac{h}{2} \qquad y_r = \frac{h}{3}$$

Summensatz:

$$x_S = \frac{A_l}{A}\,x_l + \frac{A_m}{A}\,x_m + \frac{A_r}{A}\,x_r$$

$$x_S = \frac{p}{a+c}\,\frac{2}{3}p + \frac{2c}{a+c}\,\frac{2p+c}{2} + \frac{q}{a+c}\,\frac{3p+3c+q}{3}$$

$$x_S = \frac{p}{a+c}\,\frac{2}{3}p + \frac{c}{a+c}\,2p+c + \frac{q}{a+c}\,\frac{3p+3c+q}{3}$$

$$x_S = \frac{p}{a+c}\,\frac{2}{3}p + \frac{3c}{a+c}\,\frac{2p+c}{3} + \frac{q}{a+c}\,\frac{3p+3c+q}{3}$$

$$x_S = \frac{2p^2 + 6pc + 3c^2 + 3pq + 3qc + q^2}{3(a+c)}$$

$$x_S = \frac{p^2 + c^2 + q^2 + 2pc + 2pq + 2cq + 2c^2 + p^2 + 4pc + pq + cq}{3(a+c)}$$

$$x_S = \frac{(p+c+q)^2 + 2c^2 + p^2 + 4pc + pq + cq}{3(a+c)}$$

$$x_S = \frac{(p+c+q)^2 + p^2 + 2pc + c^2 + c^2 + 2pc + pq + cq}{3(a+c)}$$

$$x_S = \frac{a^2 + (p+c)^2 + c^2 + 2pc + (p+c)q}{3(a+c)}$$

$$x_S = \frac{a^2 + (p+c)(p+c+q) + c^2 + 2pc}{3(a+c)}$$

$$x_S = \frac{a^2 + (p+c)a + c^2 + 2pc}{3(a+c)}$$

$$x_S = \frac{a^2 - c^2 + (p+c)a + 2c^2 + 2pc}{3(a+c)}$$

$$x_S = \frac{a^2 - c^2 + (p+c)a + 2(p+c)c}{3(a+c)}$$

$$\boxed{x_S = \frac{a^2 - c^2 + (p+c)(a+2c)}{3(a+c)}}$$

Summensatz:

$$y_S = \frac{A_l}{A}\, y_l + \frac{A_m}{A}\, y_m + \frac{A_r}{A}\, y_r$$

$$y_S = \frac{p}{a+c}\,\frac{h}{3} + \frac{2c}{a+c}\,\frac{h}{2} + \frac{q}{a+c}\,\frac{h}{3}$$

$$y_S = \frac{p}{a+c}\,\frac{h}{3} + \frac{3c}{a+c}\,\frac{h}{3} + \frac{q}{a+c}\,\frac{h}{3}$$

$$y_S = \frac{p + 3c + q}{a+c}\,\frac{h}{3}$$

$$\boxed{y_S = \frac{a+2c}{a+c}\,\frac{h}{3}}$$