Der Vier-Kreise-Satz von Descartes (in elementarer Darstellung)

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Vier-Kreise-Satz von Descartes

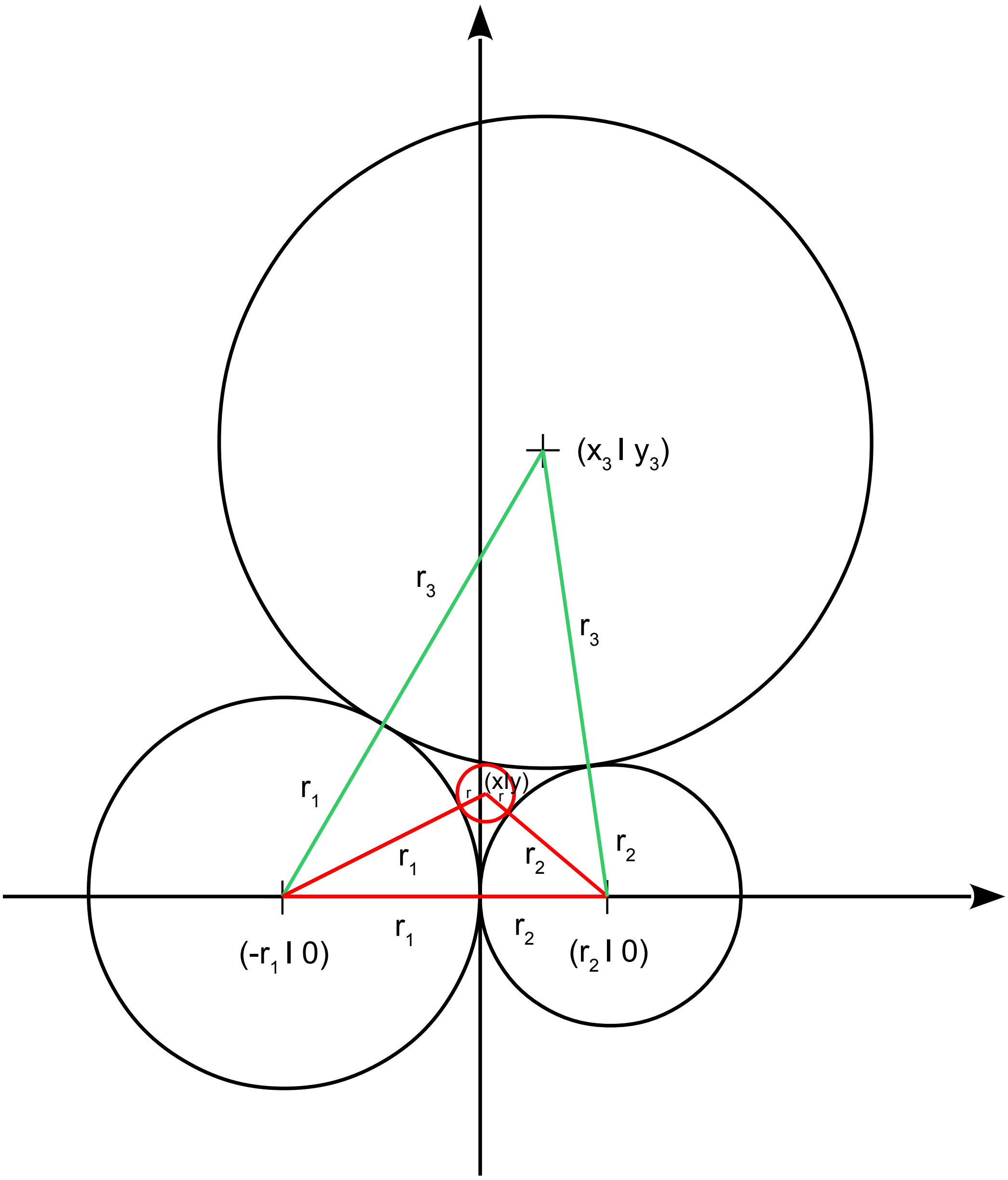
Gegeben :

3 paarweise sich berührende Kreise

$K_1 \;:=\; K_{(-r_1|0);r_1}$ $\qquad K_2 \;:=\; K_{(r_2|0);r_2}$ $\qquad K_3 \;:=\; K_{(x_3|y_3);r_3}$

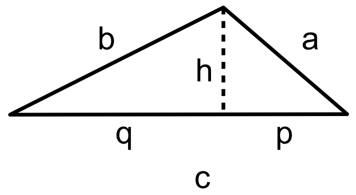
Gesucht :

Kreis $K \;:=\; K_{(x|y);r}$, der außerhalb der gegebenen Kreise liegt (und diese also von außen berührt) .



Vorbemerkung :

Im Dreieck $\Delta\, ABC$ gilt :

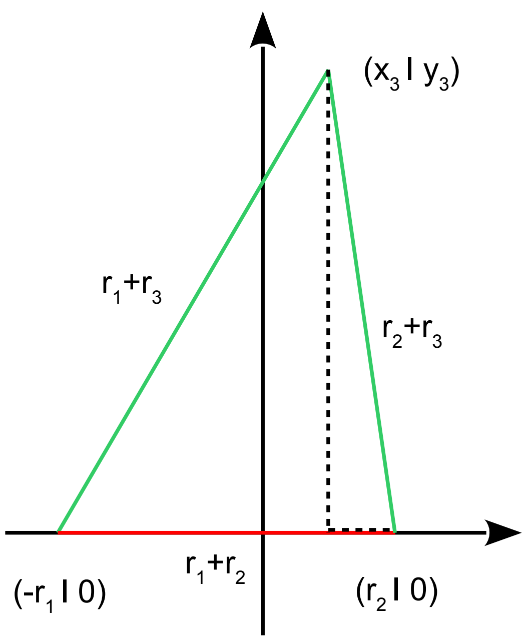


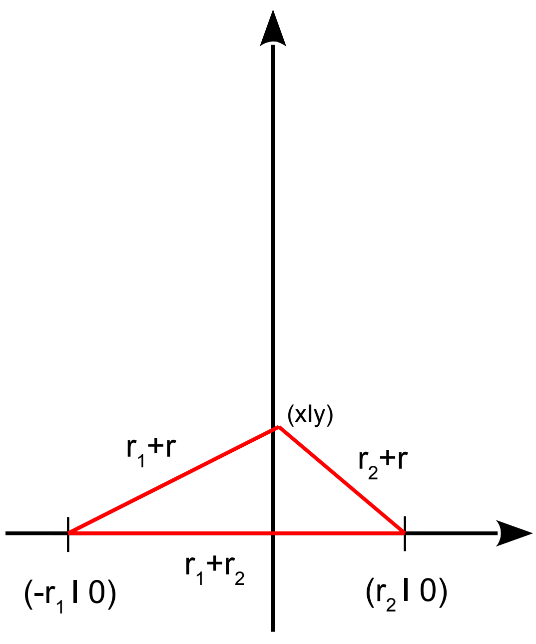
$$\boxed{p \;=\; \frac{c^2-b^2+a^2}{2c}} \quad \text{und} \quad \boxed{h \;=\; \frac{\sqrt{4c^2a^2-\left(c^2-b^2+a^2\right)^2}}{2c}}$$

.

Nun wendet man diese Formeln auf das gegeben Problem an :

Darstellung von
$(x_3|y_3)$ **in Abhängigkeit von** r_1 , r_2 , r_3 ,
$(x|y)$ **in Abhängigkeit von** r_1 , r_2 , r





Mit $a = r_2+r_3$, $b = r_1+r_3$, $c = r_1+r_2$ und $p = r_2-x_3$ $h = y_3$ folgt

$$p = \frac{c^2-b^2+a^2}{2c}$$

$$r_2-x_3 = \frac{(r_1+r_2)^2-(r_1+r_3)^2+(r_2+r_3)^2}{2(r_1+r_2)}$$

$$x_3 = r_2-\frac{(r_1+r_2)^2-(r_1+r_3)^2+(r_2+r_3)^2}{2(r_1+r_2)}$$

$$x_3 = \frac{2r_2(r_1+r_2)-(r_1+r_2)^2+(r_1+r_3)^2-(r_2+r_3)^2}{2(r_1+r_2)}$$

$$x_3 = \frac{2r_1r_2+2r_2{}^2-r_1{}^2-2r_1r_2-r_2{}^2+r_1{}^2+2r_1r_3+r_3{}^2-r_2{}^2-2r_2r_3-r_3{}^2}{2(r_1+r_2)}$$

$$x_3 = \frac{2r_1r_3-2r_2r_3}{2(r_1+r_2)}$$

$$x_3 = \frac{r_1r_3-r_2r_3}{r_1+r_2}$$

$$\boxed{x_3 = \frac{(r_1-r_2)r_3}{r_1+r_2}}$$

$$h = \frac{\sqrt{4c^2a^2-(c^2-b^2+a^2)^2}}{2c}$$

$$y_3 = \frac{\sqrt{4(r_1+r_2)^2(r_2+r_3)^2-\left((r_1+r_2)^2-(r_1+r_3)^2+(r_2+r_3)^2\right)^2}}{2(r_1+r_2)}$$

$$y_3 = \frac{\sqrt{4\left((r_1+r_2)(r_2+r_3)\right)^2-\left(r_1^2+2r_1r_2+r_2^2-r_1^2-2r_1r_3-r_3^2+r_2^2+2r_2r_3+r_3^2\right)^2}}{2(r_1+r_2)}$$

$$y_3 = \frac{\sqrt{4\left(r_1r_2+r_1r_3+r_2{}^2+r_2r_3\right)^2-\left(2r_1r_2+r_2^2-2r_1r_3+r_2^2+2r_2r_3\right)^2}}{2(r_1+r_2)}$$

$$y_3 = \frac{\sqrt{4\left(r_1r_2+r_1r_3+r_2{}^2+r_2r_3\right)^2-\left(2r_1r_2+2r_2^2-2r_1r_3+2r_2r_3\right)^2}}{2(r_1+r_2)}$$

$$y_3 = \frac{\sqrt{4\left(r_1r_2+r_1r_3+r_2{}^2+r_2r_3\right)^2-4\left(r_1r_2+r_2^2-r_1r_3+r_2r_3\right)^2}}{2(r_1+r_2)}$$

$$y_3 = \frac{\sqrt{4\left(r_1r_2+r_2{}^2+r_2r_3+r_1r_3\right)^2-4\left(r_1r_2+r_2^2+r_2r_3-r_1r_3\right)^2}}{2(r_1+r_2)}$$

$$y_3 = \frac{2\sqrt{\left(r_1r_2+r_2{}^2+r_2r_3+r_1r_3\right)^2-\left(r_1r_2+r_2^2+r_2r_3-r_1r_3\right)^2}}{2(r_1+r_2)}$$

$$y_3 = \frac{\sqrt{\left(r_1 r_2+r_2{}^2+r_2 r_3+r_1 r_3\right)^2-\left(r_1 r_2+r_2^2+r_2 r_3-r_1 r_3\right)^2}}{\left(r_1+r_2\right)}$$

$$y_3 = \frac{\sqrt{2\left(r_1 r_2+r_2{}^2+r_2 r_3\right)2\left(r_1 r_3\right)}}{\left(r_1+r_2\right)}$$

$$y_3 = \frac{2\sqrt{\left(r_1 r_2+r_2{}^2+r_2 r_3\right)\left(r_1 r_3\right)}}{\left(r_1+r_2\right)}$$

$$\boxed{y_3 = \frac{2\sqrt{\left(\left(r_1+r_2\right)r_2+r_2 r_3\right)r_1 r_3}}{\left(r_1+r_2\right)} = \frac{2\sqrt{r_1 r_2 r_3\left(r_1+r_2+r_3\right)}}{\left(r_1+r_2\right)}}$$

Es sind also

$$\boxed{x_3 = \frac{\left(r_1-r_2\right)r_3}{r_1+r_2}} \quad \text{und} \quad \boxed{y_3 = \frac{2\sqrt{\left(\left(r_1+r_2\right)r_2+r_2 r_3\right)r_1 r_3}}{\left(r_1+r_2\right)} = \frac{2\sqrt{r_1 r_2 r_3\left(r_1+r_2+r_3\right)}}{\left(r_1+r_2\right)}} \quad ,$$

und analog bekommt man

$$x = \frac{\left(r_1-r_2\right)r}{r_1+r_2} \quad \text{und} \quad y = \frac{2\sqrt{\left(\left(r_1+r_2\right)r_2+r_2 r\right)r_1 r}}{\left(r_1+r_2\right)} = \frac{2\sqrt{r_1 r_2 r^2 + r_1 r_2\left(r_1+r_2\right)r}}{\left(r_1+r_2\right)} \quad .$$

Abstandsbedingung für $\left(x|y\right)$, $\left(x_3|y_3\right)$ **:**

$$x-x_3 = \frac{\left(r_1-r_2\right)r}{r_1+r_2} - \frac{\left(r_1-r_2\right)r_3}{r_1+r_2}$$

$$x-x_3 = \frac{\left(r_1-r_2\right)r - \left(r_1-r_2\right)r_3}{r_1+r_2}$$

$$x-x_3 = \frac{\left(r_1-r_2\right)\left(r-r_3\right)}{r_1+r_2}$$

$$y-y_3 = \frac{2\sqrt{\left(\left(r_1+r_2\right)r_2+r_2 r\right)r_1 r}}{\left(r_1+r_2\right)} - \frac{2\sqrt{\left(\left(r_1+r_2\right)r_2+r_2 r_3\right)r_1 r_3}}{\left(r_1+r_2\right)}$$

$$y-y_3 = \frac{2}{r_1+r_2}\left(2\sqrt{r_1 r_2 r^2 + r_1 r_2\left(r_1+r_2\right)r} - \sqrt{r_1 r_2 r_3\left(r_1+r_2+r_3\right)}\right)$$

$$(x-x_3)^2 + (y-y_3)^2 = (r_3+r)^2$$

$$\left(\frac{(r_1-r_2)(r-r_3)}{r_1+r_2}\right)^2 + \left(\frac{2}{r_1+r_2}\left(\sqrt{r_1r_2r^2 + r_1r_2(r_1+r_2)r} - \sqrt{r_1r_2r_3(r_1+r_2+r_3)}\right)\right)^2 = (r_3+r)^2$$

$$(r_1-r_2)^2(r-r_3)^2 + 4\left(\sqrt{r_1r_2r^2 + r_1r_2(r_1+r_2)r} - \sqrt{r_1r_2r_3(r_1+r_2+r_3)}\right)^2 = (r_1+r_2)^2(r_3+r)^2$$

$$4\left(\sqrt{r_1r_2r^2 + r_1r_2(r_1+r_2)r} - \sqrt{r_1r_2r_3(r_1+r_2+r_3)}\right)^2 = (r_1+r_2)^2(r_3+r)^2 - (r_1-r_2)^2(r-r_3)^2$$

$$4\left(\sqrt{r_1r_2r^2 + r_1r_2(r_1+r_2)r} - \sqrt{r_1r_2r_3(r_1+r_2+r_3)}\right)^2 = \left((r_1+r_2)(r_3+r)\right)^2 - \left((r_1-r_2)(r-r_3)\right)^2$$

$$4\left(\sqrt{r_1r_2r^2 + r_1r_2(r_1+r_2)r} - \sqrt{r_1r_2r_3(r_1+r_2+r_3)}\right)^2 = (r_1r_3+r_1r+r_2r_3+r_2r)^2-(r_1r-r_1r_3-r_2r+r_2r_3)^2$$

$$4\left(\sqrt{r_1r_2r^2 + r_1r_2(r_1+r_2)r} - \sqrt{r_1r_2r_3(r_1+r_2+r_3)}\right)^2 = \left(r_1r+r_2r_3+(r_1r_3+r_2r)\right)^2-\left(r_1r+r_2r_3-(r_1r_3+r_2r)\right)^2$$

$$4\left(\sqrt{r_1r_2r^2 + r_1r_2(r_1+r_2)r} - \sqrt{r_1r_2r_3(r_1+r_2+r_3)}\right)^2 = 2(r_1r+r_2r_3)2(r_1r_3+r_2r)$$

$$4\left(\sqrt{r_1r_2r^2 + r_1r_2(r_1+r_2)r} - \sqrt{r_1r_2r_3(r_1+r_2+r_3)}\right)^2 = 4\left(r_1r_2r^2 + (r_1^2r_3+r_2^2r_3)r + r_1r_2r_3^2\right)$$

$$\left(\sqrt{r_1r_2r^2 + r_1r_2(r_1+r_2)r} - \sqrt{r_1r_2r_3(r_1+r_2+r_3)}\right)^2 = r_1r_2r^2 + (r_1^2r_3+r_2^2r_3)r + r_1r_2r_3^2$$

$$r_1r_2r^2+r_1r_2(r_1+r_2)r-2\sqrt{r_1r_2r^2+r_1r_2(r_1+r_2)r}\ \sqrt{r_1r_2r_3(r_1+r_2+r_3)}+r_1r_2r_3(r_1+r_2+r_3) = r_1r_2r^2+(r_1^2r_3+r_2^2r_3)r+r_1r_2r_3^2$$

$$r_1r_2(r_1+r_2)r-2\sqrt{r_1r_2r^2+r_1r_2(r_1+r_2)r}\ \sqrt{r_1r_2r_3(r_1+r_2+r_3)}+r_1r_2r_3(r_1+r_2) = (r_1^2r_3+r_2^2r_3)r$$

$$\left[r_1r_2(r_1+r_2) - (r_1^2r_3+r_2^2r_3)\right]r + r_1r_2r_3(r_1+r_2) = 2\sqrt{r_1r_2r^2+r_1r_2(r_1+r_2)r}\ \sqrt{r_1r_2r_3(r_1+r_2+r_3)}$$

$$\left(\left[r_1r_2(r_1+r_2) - (r_1^2r_3+r_2^2r_3)\right]r + r_1r_2r_3(r_1+r_2)\right)^2 = 4\left(r_1r_2r_3(r_1+r_2+r_3)\right)\left(r_1r_2r^2 + r_1r_2(r_1+r_2)r\right)$$

Quadratische Gleichung in r :

$\left[r_1r_2(r_1+r_2) - (r_1^2r_3+r_2^2r_3)\right]^2r^2$	$=$	$4r_1^2r_2^2r_3(r_1+r_2+r_3)r^2$
$+$		$+$
$2r_1r_2r_3(r_1+r_2)\left(r_1r_2(r_1+r_2) - (r_1^2r_3+r_2^2r_3)\right)r$		$4r_1^2r_2^2r_3(r_1+r_2)(r_1+r_2+r_3)r$
$+$		
$r_1^2r_2^2r_3^2(r_1+r_2)^2$		

Jetzt bringt man die Quadratische Gleichung in r auf Normalform :

Koeffizient von r^2 :

$$\left[r_1r_2(r_1+r_2) - (r_1^2r_3+r_2^2r_3)\right]^2 - 4r_1^2r_2^2r_3(r_1+r_2+r_3)$$

$$\left[r_1r_2(r_1+r_2) - r_3(r_1^2+r_2^2)\right]^2 - 4r_1^2r_2^2r_3(r_1+r_2+r_3)$$

Der Term $2r_1r_2r_3$ wird hinzugefügt und weggenommen :

$$\left[r_1r_2(r_1+r_2) - r_3(r_1^2+2r_1r_2+r_2^2) + 2r_1r_2r_3\right]^2 - 4r_1^2r_2^2r_3(r_1+r_2+r_3)$$
$$\left[r_1r_2(r_1+r_2) - r_3(r_1+r_2)^2 + 2r_1r_2r_3\right]^2 - 4r_1^2r_2^2r_3(r_1+r_2+r_3)$$
$$r_1^2r_2^2(r_1+r_2)^2+r_3^2(r_1+r_2)^4+4r_1^2r_2^2r_3^2-2r_1r_2r_3(r_1+r_2)^3+4r_1^2r_2^2r_3(r_1+r_2)-4r_1r_2r_3^2(r_1+r_2)^2 - 4r_1^2r_2^2r_3(r_1+r_2+r_3)$$
$$r_1^2r_2^2(r_1+r_2)^2+r_3^2(r_1+r_2)^4-2r_1r_2r_3(r_1+r_2)^3-4r_1r_2r_3^2(r_1+r_2)^2$$
$$(r_1+r_2)^2\left(r_1^2r_2^2 + r_3^2(r_1+r_2)^2 - 2r_1r_2r_3(r_1+r_2) - 4r_1r_2r_3^2\right)$$
$$(r_1+r_2)^2\left(r_1^2r_2^2 + r_1^2r_3^2 + 2r_1r_2r_3^2 + r_2^2r_3^2 - 2r_1^2r_2r_3 - 2r_1r_2^2r_3 - 4r_1r_2r_3^2\right)$$
$$(r_1+r_2)^2\left(r_1^2r_2^2 + r_1^2r_3^2 - 2r_1r_2r_3^2 + r_2^2r_3^2 - 2r_1^2r_2r_3 - 2r_1r_2^2r_3\right)$$
$$(r_1+r_2)^2\left(r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2 - 2r_1r_2r_3^2 - 2r_1^2r_2r_3 - 2r_1r_2^2r_3\right)$$

$$\boxed{(r_1+r_2)^2\ \left(r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2 - 2r_1^2r_2r_3 - 2r_1r_2^2r_3 - 2r_1r_2r_3^2\right)}$$ Koeffizient von r^2

Koeffizient von r :

$$2r_1r_2r_3(r_1+r_2)\left(r_1r_2(r_1+r_2) - (r_1^2r_3+r_2^2r_3)\right) - 4r_1^2r_2^2r_3(r_1+r_2)(r_1+r_2+r_3)$$

$$2r_1r_2r_3(r_1+r_2)\ \left(r_1r_2(r_1+r_2) - (r_1^2r_3+r_2^2r_3) - 2r_1r_2(r_1+r_2+r_3)\right)$$
$$2r_1r_2r_3(r_1+r_2)\ \left(r_1^2r_2 + r_1r_2^2 - r_1^2r_3-r_2^2r_3 - 2r_1^2r_2 - 2r_1r_2^2 - 2r_1r_2r_3\right)$$
$$2r_1r_2r_3(r_1+r_2)\ \left(-r_1^2r_2 - r_1r_2^2 - r_1^2r_3 - r_2^2r_3 - 2r_1r_2r_3\right)$$
$$2r_1r_2r_3(r_1+r_2)\ \left(-r_1^2r_2 - r_1r_2^2 - r_1^2r_3 - r_2^2r_3 - r_1r_2r_3 - r_1r_2r_3\right)$$
$$2r_1r_2r_3(r_1+r_2)\ \left(-r_1r_2(r_1+r_2) - r_1r_3(r_1+r_2) - r_2r_3(r_1+r_2)\right)$$

$$\boxed{-2r_1r_2r_3(r_1+r_2)^2\ \left(r_1r_2 + r_1r_3 + r_2r_3\right)}$$ Koeffizient von r

Absolutglied :

$$\boxed{r_1^2r_2^2r_3^2\ (r_1+r_2)^2}$$

Die quadratische Gleichung in r in Normalform lautet nun

$$\left(r_1+r_2\right)^2\left(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2-2r_1^2r_2r_3-2r_1r_2^2r_3-2r_1r_2r_3^2\right)r^2 \;-\; 2r_1r_2r_3\left(r_1+r_2\right)^2\left(r_1r_2+r_1r_3+r_2r_3\right)r \;+\; r_1^2r_2^2r_3^2\left(r_1+r_2\right)^2 \;=\; 0$$

$$\left(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2-2r_1^2r_2r_3-2r_1r_2^2r_3-2r_1r_2r_3^2\right)r^2 \;-\; 2r_1r_2r_3\left(r_1r_2+r_1r_3+r_2r_3\right)r \;+\; r_1^2r_2^2r_3^2 \;=\; 0 \quad .$$

Formulierung der Gleichung über die Krümmung $\dfrac{1}{r}$:

$$\left(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2-2r_1^2r_2r_3-2r_1r_2^2r_3-2r_1r_2r_3^2\right) \;-\; 2r_1r_2r_3\left(r_1r_2+r_1r_3+r_2r_3\right)\left(\frac{1}{r}\right) \;+\; r_1^2r_2^2r_3^2\left(\frac{1}{r}\right)^2 \;=\; 0$$

$$r_1^2r_2^2r_3^2\left(\frac{1}{r}\right)^2 \;-\; 2r_1r_2r_3\left(r_1r_2+r_1r_3+r_2r_3\right)\left(\frac{1}{r}\right) \;+\; \left(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2-2r_1^2r_2r_3-2r_1r_2^2r_3-2r_1r_2r_3^2\right) \;=\; 0$$

$$\left(\frac{1}{r}\right)^2 \;-\; \frac{2\left(r_1r_2+r_1r_3+r_2r_3\right)}{r_1r_2r_3}\left(\frac{1}{r}\right) \;+\; \frac{r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2-2r_1^2r_2r_3-2r_1r_2^2r_3-2r_1r_2r_3^2}{r_1^2r_2^2r_3^2} \;=\; 0$$

$$\left(\frac{1}{r}\right)^2 \;-\; \frac{2\left(r_1r_2+r_1r_3+r_2r_3\right)}{r_1r_2r_3}\frac{1}{r} \;+\; \frac{r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2-2r_1^2r_2r_3-2r_1r_2^2r_3-2r_1r_2r_3^2}{r_1^2r_2^2r_3^2} \;=\; 0$$

$$\left(\frac{1}{r}\right)^2 \;-\; 2\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r}\right) \;+\; \frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}-2\left(\frac{1}{r_1}\frac{1}{r_2}+\frac{1}{r_1}\frac{1}{r_3}+\frac{1}{r_2}\frac{1}{r_3}\right) \;=\; 0$$

Die Lösungen sind

$$\frac{1}{r} \;=\; \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \;\pm\; \sqrt{\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2 \;-\; \frac{1}{r_1^2}-\frac{1}{r_2^2}-\frac{1}{r_3^2} \;+\; 2\left(\frac{1}{r_1}\frac{1}{r_2}+\frac{1}{r_1}\frac{1}{r_3}+\frac{1}{r_2}\frac{1}{r_3}\right)}$$

$$\frac{1}{r} \;=\; \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \;\pm\; \sqrt{\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2} \;+\; 2\left(\frac{1}{r_1}\frac{1}{r_2}+\frac{1}{r_1}\frac{1}{r_3}+\frac{1}{r_2}\frac{1}{r_3}\right) \;-\; \frac{1}{r_1^2}-\frac{1}{r_2^2}-\frac{1}{r_3^2} \;+\; 2\left(\frac{1}{r_1}\frac{1}{r_2}+\frac{1}{r_1}\frac{1}{r_3}+\frac{1}{r_2}\frac{1}{r_3}\right)}$$

$$\frac{1}{r} \;=\; \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \;\pm\; \sqrt{4\left(\frac{1}{r_1}\frac{1}{r_2}+\frac{1}{r_1}\frac{1}{r_3}+\frac{1}{r_2}\frac{1}{r_3}\right)}$$

$$\frac{1}{r} \;=\; \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \;\pm\; 2\sqrt{\frac{1}{r_1}\frac{1}{r_2}+\frac{1}{r_1}\frac{1}{r_3}+\frac{1}{r_2}\frac{1}{r_3}}$$

$$r \;=\; \frac{1}{\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3} \;\pm\; 2\sqrt{\dfrac{1}{r_1}\dfrac{1}{r_2}+\dfrac{1}{r_1}\dfrac{1}{r_3}+\dfrac{1}{r_2}\dfrac{1}{r_3}}}$$

Durch Einsetzen ergeben sich die Koordinaten des Mittelpunktes :

$$\boxed{x \;=\; \frac{\left(r_1-r_2\right)r}{r_1+r_2}} \quad , \quad \boxed{y \;=\; \frac{2\sqrt{\left(\left(r_1+r_2\right)r_2+r_2r\right)r_1r}}{\left(r_1+r_2\right)} \;=\; \frac{2\sqrt{\left(r_1+r_2+r\right)r_1r_2r}}{\left(r_1+r_2\right)} \;=\; \frac{2\sqrt{r_1r_2r^2 \;+\; r_1r_2\left(r_1+r_2\right)r}}{\left(r_1+r_2\right)}}$$

mit $r \;=\; \dfrac{1}{\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3} \;\pm\; 2\sqrt{\dfrac{1}{r_1}\dfrac{1}{r_2}+\dfrac{1}{r_1}\dfrac{1}{r_3}+\dfrac{1}{r_2}\dfrac{1}{r_3}}}$

Ersetzt man die Radien durch die Krümmungen , $k_1=\frac{1}{r_1}$, $k_2=\frac{1}{r_2}$, $k_3=\frac{1}{r_3}$, $k=\frac{1}{r}$,
so erhält man die quadratische Gleichung

$$k^2 - 2(k_1+k_2+k_3)k + k_1^2+k_2^2+k_3^2-2(k_1k_2+k_1k_3+k_2k_3) = 0 \quad .$$

Die Lösungen sind

$$k = k_1+k_2+k_3 \pm \sqrt{(k_1+k_2+k_3)^2 - k_1^2-k_2^2-k_3^2 + 2(k_1k_2+k_1k_3+k_2k_3)}$$

$$k = k_1+k_2+k_3 \pm \sqrt{k_1^2+k_2^2+k_3^2 + 2(k_1k_2+k_1k_3+k_2k_3) - k_1^2-k_2^2-k_3^2 + 2(k_1k_2+k_1k_3+k_2k_3)}$$

$$k = k_1+k_2+k_3 \pm \sqrt{4(k_1k_2+k_1k_3+k_2k_3)}$$

$$k = k_1+k_2+k_3 \pm 2\sqrt{(k_1k_2+k_1k_3+k_2k_3)}$$

Umformung der Gleichung liefert :

$$k^2 - 2(k_1+k_2+k_3)k + k_1^2+k_2^2+k_3^2-2(k_1k_2+k_1k_3+k_2k_3) = 0$$

$$k_1^2+k_2^2+k_3^2+k^2 - 2(k_1+k_2+k_3)k-2(k_1k_2+k_1k_3+k_2k_3) = 0$$

$$k_1^2+k_2^2+k_3^2+k^2 = 2(k_1k_2+k_1k_3+k_2k_3) + 2(k_1+k_2+k_3)k$$

$$k_1^2+k_2^2+k_3^2+k^2 = 2(k_1k_2+k_1k_3+k_2k_3 + (k_1+k_2+k_3)k)$$

$$k_1^2+k_2^2+k_3^2+k^2 = 2(k_1k_2+k_1k_3+k_2k_3 + k_1k+k_2k+k_3k)$$

$$k_1^2+k_2^2+k_3^2+k^2 = 2(k_1k_2+k_1k_3+k_1k + k_2k_3+k_2k + k_3k)$$

$$2(k_1^2+k_2^2+k_3^2+k^2) = k_1^2+k_2^2+k_3^2+k^2 + 2(k_1k_2+k_1k_3+k_1k + k_2k_3+k_2k + k_3k)$$

$$\boxed{2(k_1^2 + k_2^2 + k_3^2 + k^2) = (k_1 + k_2 + k_3 + k)^2}$$

Vier-Kreise-Satz von Descartes (1596 – 1650)