Descartes‘ Vier-Kreise-Satz und komplexer Vier-Kreise-Satz

(nach Burnett-Stuart)

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Bemerkung :

Beweise für den Vier-Kreise-Satz und den komplexen Vierkreise-Satz kann man im Internet nur selten finden, und wenn, allenfalls in mehr oder weniger schwieriger Darstellung.

Ziel dieses Skriptums ist es , die Beweise von **Burnett-Stuart** für den Vier-Kreise-Satz und den komplexen Vier-Kreise-Satz ausführlicher und übersichtlicher und damit etwas verständlicher darzustellen.

Quellen :

Burnett-Stuart, George (2014) : A Proof of Descartes' Circle Theorem (concerning four circles, each of which touches the remaining three)
http://foothills-ts.net/mother/DescartesCircleTheorem.htm

Burnett-Stuart, George (2014): A Proof of the Extended Descartes' Circle Theorem
http://foothills-ts.net/mother/ExtendedDescartesCircleTheorem.htm

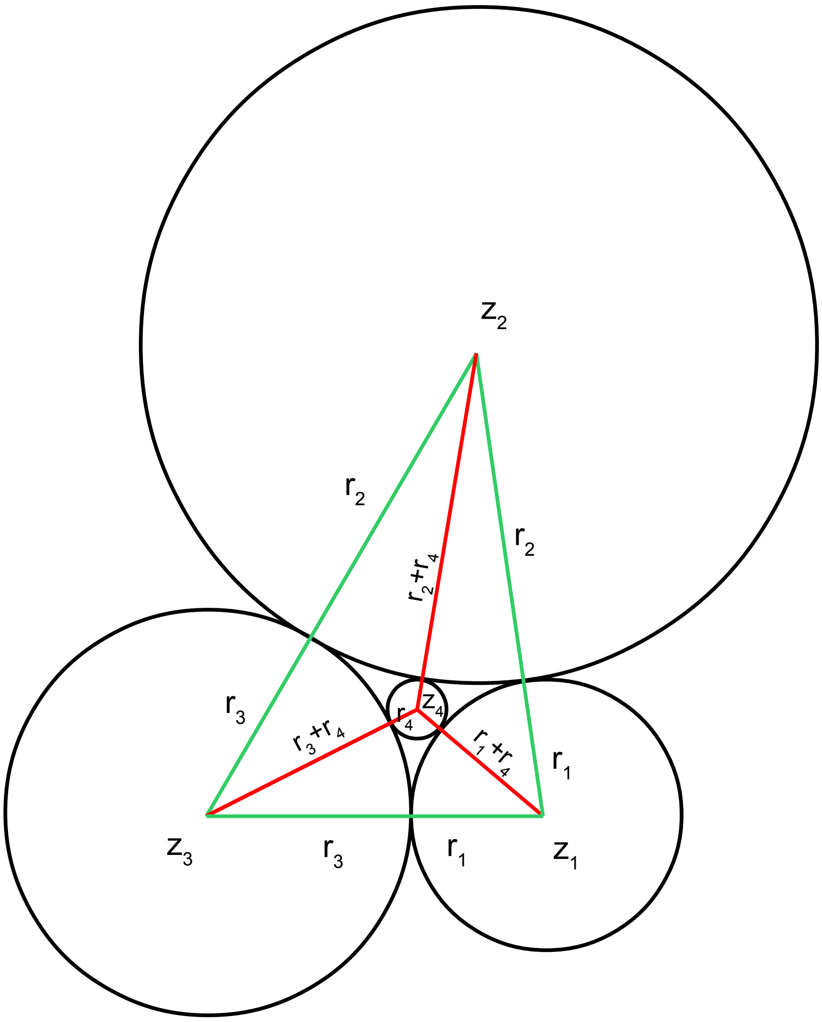
Fehringer, Arno (2018) : Der Vier-Kreise-Satz von Descartes (in elementarer Darstellung)
http://mathematikgarten.npage.de/get_file.php?id=33078777&vnr=385360

Der Vier-Kreise-Satz von Descartes

Gegeben sind vier paarweise berührende Kreise K_i mit Zentren z_i und Radien r_i , $i \in \{1;2;3:4\}$.

Dann gilt die Gleichung $\boxed{2\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}\right) = \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right)^2}$.

Beweis :



Winkel :

$\varphi_{12} := \sphericalangle z_1 z_4 z_2$, $\varphi_{23} := \sphericalangle z_2 z_4 z_3$, $\varphi_{31} := \sphericalangle z_3 z_4 z_1$

Kosinussatz :

$$\left(r_1+r_2\right)^2 = \left(r_1+r_4\right)^2 + \left(r_2+r_4\right)^2 - 2\left(r_1+r_4\right)\left(r_2+r_4\right)\cos\varphi_{12}$$

$$\cos\varphi_{12} = \frac{\left(r_1+r_4\right)^2 + \left(r_2+r_4\right)^2 - \left(r_1+r_2\right)^2}{2\left(r_1+r_4\right)\left(r_2+r_4\right)}$$

$$\cos\varphi_{12} = \frac{r_1^2+2r_1r_4+r_4^2 + r_2^2+2r_2r_4+r_4^2 - r_1^2-2r_1r_2-r_2^2}{2\left(r_1+r_4\right)\left(r_2+r_4\right)}$$

$$\cos\varphi_{12} = \frac{2r_1r_4 + r_4^2 + 2r_2r_4 + r_4^2 - 2r_1r_2}{2\left(r_1+r_4\right)\left(r_2+r_4\right)}$$

$$\cos\varphi_{12} = \frac{2r_4^2 + 2r_1r_4 + 2r_2r_4 - 2r_1r_2}{2\left(r_1+r_4\right)\left(r_2+r_4\right)}$$

$$\cos\varphi_{12} = \frac{2r_4^2 + 2r_1r_4 + 2r_2r_4 + 2r_1r_2 - 4r_1r_2}{2\left(r_1+r_4\right)\left(r_2+r_4\right)}$$

$$\cos\varphi_{12} = \frac{2\left(r_1+r_4\right)\left(r_2+r_4\right) - 4r_1r_2}{2\left(r_1+r_4\right)\left(r_2+r_4\right)}$$

$$\boxed{\cos\varphi_{12} = 1 - 2\frac{r_1r_2}{\left(r_1+r_4\right)\left(r_2+r_4\right)}} \qquad \textbf{(1)}$$

Durch zyklische Vertauschung der Indizes erhält man analog :

$$\boxed{\cos\varphi_{23} = 1 - 2\frac{r_2r_3}{\left(r_2+r_4\right)\left(r_3+r_4\right)}} \qquad \textbf{(2)}$$

$$\boxed{\cos\varphi_{31} = 1 - 2\frac{r_3r_1}{\left(r_3+r_4\right)\left(r_1+r_4\right)}} \qquad \textbf{(3)}$$

Additionstheorem für cosinus :

$$\cos\varphi_{12} = \cos\left(\frac{\varphi_{12}}{2}+\frac{\varphi_{12}}{2}\right)$$

$$\cos\varphi_{12} = \cos^2\frac{\varphi_{12}}{2} - \sin^2\frac{\varphi_{12}}{2}$$

$$\cos\varphi_{12} = 1-\sin^2\frac{\varphi_{12}}{2} - \sin^2\frac{\varphi_{12}}{2}$$

$$\cos\varphi_{12} = 1-2\sin^2\frac{\varphi_{12}}{2}$$

$$2\sin^2\frac{\varphi_{12}}{2} = 1-\cos\varphi_{12}$$
$$\sin^2\frac{\varphi_{12}}{2} = \frac{1-\cos\varphi_{12}}{2}$$
$$\sin\frac{\varphi_{12}}{2} = \sqrt{\frac{1-\cos\varphi_{12}}{2}}$$
$$\sin\frac{\varphi_{12}}{2} = \sqrt{\frac{1}{2}-\frac{1}{2}\cos\varphi_{12}}$$
$$\sin\frac{\varphi_{12}}{2} = \sqrt{\frac{1}{2}-\frac{1}{2}\left(1 - 2\frac{r_1 r_2}{(r_1+r_4)(r_2+r_4)}\right)} \qquad \text{wegen (1)}$$

$$\sin\frac{\varphi_{12}}{2} = \sqrt{\frac{1}{2}-\frac{1}{2} + \frac{r_1 r_2}{(r_1+r_4)(r_2+r_4)}}$$

$$\boxed{\sin\frac{\varphi_{12}}{2} = \sqrt{\frac{r_1 r_2}{(r_1+r_4)(r_2+r_4)}}} \quad , \quad \boxed{\sin^2\frac{\varphi_{12}}{2} = \frac{r_1 r_2}{(r_1+r_4)(r_2+r_4)}}$$

Analog erhält man durch zyklische Vertauschung

$$\boxed{\sin\frac{\varphi_{23}}{2} = \sqrt{\frac{r_2 r_3}{(r_2+r_4)(r_3+r_4)}}} \quad , \quad \boxed{\sin^2\frac{\varphi_{23}}{2} = \frac{r_2 r_3}{(r_2+r_4)(r_3+r_4)}}$$

$$\boxed{\sin\frac{\varphi_{31}}{2} = \sqrt{\frac{r_3 r_1}{(r_3+r_4)(r_1+r_4)}}} \quad , \quad \boxed{\sin^2\frac{\varphi_{31}}{2} = \frac{r_3 r_1}{(r_3+r_4)(r_1+r_4)}}$$

Winkelsummensatz :

$$\varphi_{12} + \varphi_{23} + \varphi_{31} = 2\pi$$
$$\frac{\varphi_{12}}{2} + \frac{\varphi_{23}}{2} + \frac{\varphi_{31}}{2} = \pi$$
$$\frac{\varphi_{12}}{2} = \pi - \frac{\varphi_{23}}{2} - \frac{\varphi_{31}}{2}$$

Symmetrie von sinus und Additionstheorem :

$$\sin\frac{\varphi_{12}}{2} = \sin\left(\pi - \frac{\varphi_{23}}{2} - \frac{\varphi_{31}}{2}\right)$$
$$\sin\frac{\varphi_{12}}{2} = \sin\left(\frac{\varphi_{23}}{2} + \frac{\varphi_{31}}{2}\right)$$

$$\sin\frac{\varphi_{12}}{2} = \sin\frac{\varphi_{23}}{2}\cos\frac{\varphi_{31}}{2} + \cos\frac{\varphi_{23}}{2}\sin\frac{\varphi_{31}}{2}$$

Mit $\cos\frac{\varphi_{23}}{2} = \sqrt{1-\sin^2\frac{\varphi_{23}}{2}}$, $\cos\frac{\varphi_{31}}{2} = \sqrt{1-\sin^2\frac{\varphi_{31}}{2}}$ folgt

$$\sin\frac{\varphi_{12}}{2} = \sin\frac{\varphi_{23}}{2}\sqrt{1-\sin^2\frac{\varphi_{31}}{2}} + \sqrt{1-\sin^2\frac{\varphi_{23}}{2}}\sin\frac{\varphi_{31}}{2} \quad .$$

Setzt man vorübergehend

$$s := \sin\frac{\varphi_{12}}{2} = \sqrt{\frac{r_1 r_2}{(r_1+r_4)(r_2+r_4)}}$$

$$t := \sin\frac{\varphi_{23}}{2} = \sqrt{\frac{r_2 r_3}{(r_2+r_4)(r_3+r_4)}}$$

$$u := \sin\frac{\varphi_{31}}{2} = \sqrt{\frac{r_3 r_1}{(r_3+r_4)(r_1+r_4)}} \quad ,$$

so folgt weiter :

$$\sin\frac{\varphi_{12}}{2} = \sin\frac{\varphi_{23}}{2}\sqrt{1-\sin^2\frac{\varphi_{31}}{2}} + \sqrt{1-\sin^2\frac{\varphi_{23}}{2}}\sin\frac{\varphi_{31}}{2}$$

$$s = t\sqrt{1-u^2} + \sqrt{1-t^1}u$$
$$s^2 = t^2(1-u^2) + 2tu\sqrt{1-t^2}\sqrt{1-u^2} + (1-t^1)u^2$$
$$s^2 = t^2-t^2u^2 + 2tu\sqrt{1-t^2}\sqrt{1-u^2} + u^2-t^2u^2$$

$$s^2-t^2-u^2+2t^2u^2 = 2tu\sqrt{1-t^2}\sqrt{1-u^2}$$

$$(s^2-t^2-u^2+2t^2u^2)^2 = 4t^2u^2(1-t^2)(1-u^2)$$

$$\begin{aligned} & s^4 + t^4 + u^4 + 4t^4u^4 && = \quad 4t^2u^2 - 4t^2u^4 - 4t^4u^2 + 4t^4u^4 \\ & -2s^2t^2 - 2s^2u^2 + 4s^2t^2u^2 && \\ & +2t^2u^2 - 4t^4u^2 && \\ & -4t^2u^4 && \end{aligned}$$

$$s^4 + t^4 + u^4 - 2s^2t^2 - 2s^2u^2 + 4s^2t^2u^2 + 2t^2u^2 = 4t^2u^2$$

$$s^4 + t^4 + u^4 - 2s^2t^2 - 2s^2u^2 + 4s^2t^2u^2 = 2t^2u^2$$

$$s^4 + t^4 + u^4 - 2s^2t^2 - 2s^2u^2 - 2t^2u^2 + 4s^2t^2u^2 = 0$$

$$s^4 + t^4 + u^4 - 2(s^2t^2 + s^2u^2 + t^2u^2) + 4s^2t^2u^2 = 0$$
$$2(s^4 + t^4 + u^4) - (s^4 + t^4 + u^4) - 2(s^2t^2 + s^2u^2 + t^2u^2) + 4s^2t^2u^2 = 0$$

$$2(s^4 + t^4 + u^4) - (s^4 + t^4 + u^4 + 2(s^2t^2 + s^2u^2 + t^2u^2)) + 4s^2t^2u^2 = 0$$

$$2(s^4 + t^4 + u^4) - (s^2 + t^2 + u^2)^2 + 4s^2t^2u^2 = 0$$

$$2\left(\frac{s^2}{t^2u^2} + \frac{t^2}{u^2s^2} + \frac{u^2}{s^2t^2}\right) - \left(\frac{s}{tu} + \frac{t}{us} + \frac{u}{st}\right)^2 + 4 = 0$$

$$2\left(\left(\frac{s}{tu}\right)^2 + \left(\frac{t}{us}\right)^2 + \left(\frac{u}{st}\right)^2\right) - \left(\frac{s}{tu} + \frac{t}{us} + \frac{u}{st}\right)^2 + 4 = 0$$

Zur Rücksubstitution werden zunächst aus den Substitutionsgleichungen

$$s := \sin\frac{\varphi_{12}}{2} = \sqrt{\frac{r_1 r_2}{(r_1+r_4)(r_2+r_4)}}$$
$$t := \sin\frac{\varphi_{23}}{2} = \sqrt{\frac{r_2 r_3}{(r_2+r_4)(r_3+r_4)}}$$
$$u := \sin\frac{\varphi_{31}}{2} = \sqrt{\frac{r_3 r_1}{(r_3+r_4)(r_1+r_4)}}$$

folgende Ausdrücke dargestellt :

$$\frac{s}{tu} = \frac{\sqrt{\frac{r_1 r_2}{(r_1+r_4)(r_2+r_4)}}}{\sqrt{\frac{r_2 r_3}{(r_2+r_4)(r_3+r_4)}}\ \sqrt{\frac{r_3 r_1}{(r_3+r_4)(r_1+r_4)}}}$$

$$\frac{s}{tu} = \sqrt{\frac{\frac{r_1 r_2}{(r_1+r_4)(r_2+r_4)}}{\frac{r_2 r_3}{(r_2+r_4)(r_3+r_4)}\ \frac{r_3 r_1}{(r_3+r_4)(r_1+r_4)}}}$$

$$\frac{s}{tu} = \sqrt{\frac{r_1 r_2}{(r_1+r_4)(r_2+r_4)}\ \frac{(r_2+r_4)(r_3+r_4)}{r_2 r_3}\ \frac{(r_3+r_4)(r_1+r_4)}{r_3 r_1}}$$

$$\frac{s}{tu} = \sqrt{\frac{(r_3+r_4)^2}{r_3^2}}$$

$$\frac{s}{tu} = \frac{(r_3+r_4)}{r_3}$$

$$\frac{s}{tu} = \frac{1}{\frac{r_3}{(r_3+r_4)}}$$

Analog durch zyklische Vertauschung

$$\frac{t}{us} = \frac{1}{\frac{r_1}{(r_1+r_4)}}$$

$$\frac{u}{st} = \frac{1}{\frac{r_2}{(r_2+r_4)}}$$

Diese Ausdrücke werden nun eingesetzt in

$$2\left(\left(\frac{s}{tu}\right)^2 + \left(\frac{t}{us}\right)^2 + \left(\frac{u}{st}\right)^2\right) - \left(\frac{s}{tu} + \frac{t}{us} + \frac{u}{st}\right)^2 + 4 = 0$$

$$\boxed{2\left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)^2} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)^2} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)^2}\right) - \left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)}\right)^2 + 4 = 0} \qquad \textbf{(4)}$$

Die abschließende Umformung ergibt den Satz von Descartes :

Wegen

$$\frac{1}{\left(\frac{r_i}{(r_i+r_4)}\right)} = \frac{r_i+r_4}{r_i} = 1+\frac{r_4}{r_i} \qquad , \qquad i \in \{1:2:3\}$$

folgt :

$$2\left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)^2} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)^2} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)^2}\right) - \left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)}\right)^2 + 4 = 0$$

$$2\left(\left(1+\frac{r_4}{r_1}\right)^2 + \left(1+\frac{r_4}{r_2}\right)^2 + \left(1+\frac{r_4}{r_3}\right)^2\right) - \left(1+\frac{r_4}{r_1} + 1+\frac{r_4}{r_2} + 1+\frac{r_4}{r_3}\right)^2 + 4 = 0$$

$$2\left(1+2\frac{r_4}{r_1}+\left(\frac{r_4}{r_1}\right)^2 + 1+2\frac{r_4}{r_1}+\left(\frac{r_4}{r_1}\right)^2 + 1+2\frac{r_4}{r_1}+\left(\frac{r_4}{r_1}\right)^2\right) - \left(3 + \frac{r_4}{r_1} + \frac{r_4}{r_2} + \frac{r_4}{r_3}\right)^2 + 4 = 0$$

$$2\left(3 + 2\left(\frac{r_4}{r_1}+\frac{r_4}{r_2}+\frac{r_4}{r_3}\right) + \left(\frac{r_4}{r_1}\right)^2+\left(\frac{r_4}{r_2}\right)^2+\left(\frac{r_4}{r_3}\right)^2\right) - \left(3 + \frac{r_4}{r_1}+\frac{r_4}{r_2}+\frac{r_4}{r_3}\right)^2 + 4 = 0$$

Nach vorübergehender Substitution

$$\sigma := \left(\frac{r_4}{r_1}+\frac{r_4}{r_2}+\frac{r_4}{r_3}\right) \qquad , \qquad \tau := \left(\frac{r_4}{r_1}\right)^2+\left(\frac{r_4}{r_2}\right)^2+\left(\frac{r_4}{r_3}\right)^2$$

erhält man

$$2\left(3 + 2\left(\frac{r_4}{r_1}+\frac{r_4}{r_2}+\frac{r_4}{r_3}\right) + \left(\frac{r_4}{r_1}\right)^2+\left(\frac{r_4}{r_2}\right)^2+\left(\frac{r_4}{r_3}\right)^2\right) - \left(3 + \frac{r_4}{r_1}+\frac{r_4}{r_2}+\frac{r_4}{r_3}\right)^2 + 4 = 0$$

$$2(3 + 2\sigma + \tau) - (3 + \sigma)^2 + 4 = 0$$

$$6 + 4\sigma + 2\tau - 9 - 6\sigma - \sigma^2 + 4 = 0$$

$$1 - 2\sigma - \sigma^2 + 2\tau = 0$$

$$2\tau + 1 = \sigma^2 + 2\sigma$$

$$2\tau + 2 = \sigma^2 + 2\sigma + 1$$

$$2(\tau + 1) = (\sigma + 1)^2 \quad ,$$

und nach der Resubstitution

$$2\left(\left(\frac{r_4}{r_1}\right)^2+\left(\frac{r_4}{r_2}\right)^2+\left(\frac{r_4}{r_3}\right)^2 + 1\right) = \left(\frac{r_4}{r_1}+\frac{r_4}{r_2}+\frac{r_4}{r_3} + 1\right)^2$$

$$2\left(\frac{r_4^2}{r_1^2}+\frac{r_4^2}{r_2^2}+\frac{r_4^2}{r_3^2} + 1\right) = \left(\frac{r_4}{r_1}+\frac{r_4}{r_2}+\frac{r_4}{r_3} + 1\right)^2$$

$$2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2} + \frac{1}{r_4^2}\right) = \left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} + \frac{1}{r_4}\right)^2$$

$$\boxed{2\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}\right) = \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right)^2} \qquad \textbf{(5)}$$

Vier-Kreise-Satz von Descartes (1596 – 1650)

q.e.d.

Der komplexe Vier-Kreise-Satz von Descartes

In der komplexen Zahlenebene $\mathbb{C}$ sind vier paarweise berührende Kreise K_i mit Zentren z_i und Radien r_i , $i \in \{1;2;3;4\}$ gegeben .

Dann gilt die Gleichung $\boxed{2\left(\frac{z_1^2}{r_1^2}+\frac{z_2^2}{r_2^2}+\frac{z_3^2}{r_3^2}+\frac{z_4^2}{r_4^2}\right)=\left(\frac{z_1}{r_1}+\frac{z_2}{r_2}+\frac{z_3}{r_3}+\frac{z_4}{r_4}\right)^2}$.

Beweis :

Wenn die Gleichung richtig ist, dann muss sie für alle Koordinatensysteme richtig sein, speziell muss sie invariant gegenüber Drehungen , Streckungen des Maßstabs und Verschiebungen sein .

Bei Drehungen und Streckungen werden die Koordinaten mit einer Zahl v multipliziert. Auf beiden Seiten der Gleichung kann man deshalb den Faktor v^2 abspalten und weg dividieren, so dass man die ursprüngliche Gleichung erhält , und die Gleichung deshalb auch im neuen Koordinatensystem richtig ist.

$$2\left(\frac{vz_1^2}{r_1^2}+\frac{vz_2^2}{r_2^2}+\frac{vz_3^2}{r_3^2}+\frac{vz_4^2}{r_4^2}\right)=\left(\frac{vz_1}{r_1}+\frac{vz_2}{r_2}+\frac{vz_3}{r_3}+\frac{vz_4}{r_4}\right)^2$$

$$2\left(\frac{z_1^2}{r_1^2}+\frac{z_2^2}{r_2^2}+\frac{z_3^2}{r_3^2}+\frac{z_4^2}{r_4^2}\right)v^2=\left(\frac{z_1}{r_1}+\frac{z_2}{r_2}+\frac{z_3}{r_3}+\frac{z_4}{r_4}\right)^2v^2$$

$$2\left(\frac{z_1^2}{r_1^2}+\frac{z_2^2}{r_2^2}+\frac{z_3^2}{r_3^2}+\frac{z_4^2}{r_4^2}\right)=\left(\frac{z_1}{r_1}+\frac{z_2}{r_2}+\frac{z_3}{r_3}+\frac{z_4}{r_4}\right)^2$$

Um die Invarianz gegenüber Verschiebungen w des Koordinatensystems zu garantieren, muss für alle w gelten :

$$2\left(\frac{(z_1-w)^2}{r_1^2}+\frac{(z_2-w)^2}{r_2^2}+\frac{(z_3-w)^2}{r_3^2}+\frac{(z_4-w)^2}{r_4^2}\right)=\left(\frac{z_1-w}{r_1}+\frac{z_2-w}{r_2}+\frac{z_3-w}{r_3}+\frac{z_4-w}{r_4}\right)^2$$

$$2\left(\frac{z_1^2-2z_1w+w^2}{r_1^2}+\ldots+\frac{z_4^2-2z_4w+w^2}{r_4^2}\right)=\left(\frac{z_1-w}{r_1}+\frac{z_2-w}{r_2}+\frac{z_3-w}{r_3}+\frac{z_4-w}{r_4}\right)^2$$

$$2\left(\frac{z_1^2-2z_1w+w^2}{r_1^2}+\ldots+\frac{z_4^2-2z_4w+w^2}{r_4^2}\right)=\left(\frac{z_1}{r_1}+\frac{z_2}{r_2}+\frac{z_3}{r_3}+\frac{z_4}{r_4}-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)w\right)^2$$

$$
\begin{aligned}
&2\left(\frac{z_1^2}{r_1^2}+\frac{z_2^2}{r_2^2}+\frac{z_{31}^2}{r_3^2}+\frac{z_4^2}{r_4^2}\right) && = \left(\frac{z_1}{r_1}+\frac{z_2}{r_2}+\frac{z_3}{r_3}+\frac{z_4}{r_4}\right)^2 \\
&-\ 4\left(\frac{z_1}{r_1^2}+\frac{z_2}{r_2^2}+\frac{z_3}{r_3^2}+\frac{z_4}{r_4^2}\right)w && -\ 2\left(\frac{z_1}{r_1}+\frac{z_2}{r_2}+\frac{z_3}{r_3}+\frac{z_4}{r_4}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)w \\
&+\ 2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right)w^2 && +\ \left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2 w^2
\end{aligned}
$$

Die Koeffizienten des quadratischen Terms sind nach dem Satz von Descartes (5) gleich.

Der Koeffizientenvergleich des linearen Terms und des Absolutgliedes liefert zwei Gleichungen , deren Gültigkeit nachgewiesen werden muss :

$$
2\left(\frac{z_1}{r_1^2}+\frac{z_2}{r_2^2}+\frac{z_3}{r_3^2}+\frac{z_4}{r_4^2}\right) = \left(\frac{z_1}{r_1}+\frac{z_2}{r_2}+\frac{z_3}{r_3}+\frac{z_4}{r_4}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right) \qquad \textbf{(6)}
$$

$$
2\left(\frac{z_1^2}{r_1^2}+\frac{z_1^2}{r_1^2}+\frac{z_1^2}{r_1^2}+\frac{z_1^2}{r_1^2}\right) = \left(\frac{z_1}{r_1}+\frac{z_2}{r_2}+\frac{z_3}{r_3}+\frac{z_4}{r_4}\right)^2 \qquad \textbf{(7)}
$$

Zu Gleichung (6) :

$$
2\left(\frac{z_1}{r_1^2}+\frac{z_2}{r_2^2}+\frac{z_3}{r_3^2}+\frac{z_4}{r_4^2}\right) = \left(\frac{z_1}{r_1}+\frac{z_2}{r_2}+\frac{z_3}{r_3}+\frac{z_4}{r_4}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)
$$

$$
\begin{aligned}
&\left[\frac{2}{r_1^2} - \frac{1}{r_1}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]z_1 \qquad = \quad 0 \\
&\quad + \left[\frac{2}{r_2^2} - \frac{1}{r_2}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]z_2 \\
&\qquad + \left[\frac{2}{r_3^2} - \frac{1}{r_3}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]z_3 \\
&\qquad\quad + \left[\frac{2}{r_4^2} - \frac{1}{r_4}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]z_4
\end{aligned}
$$

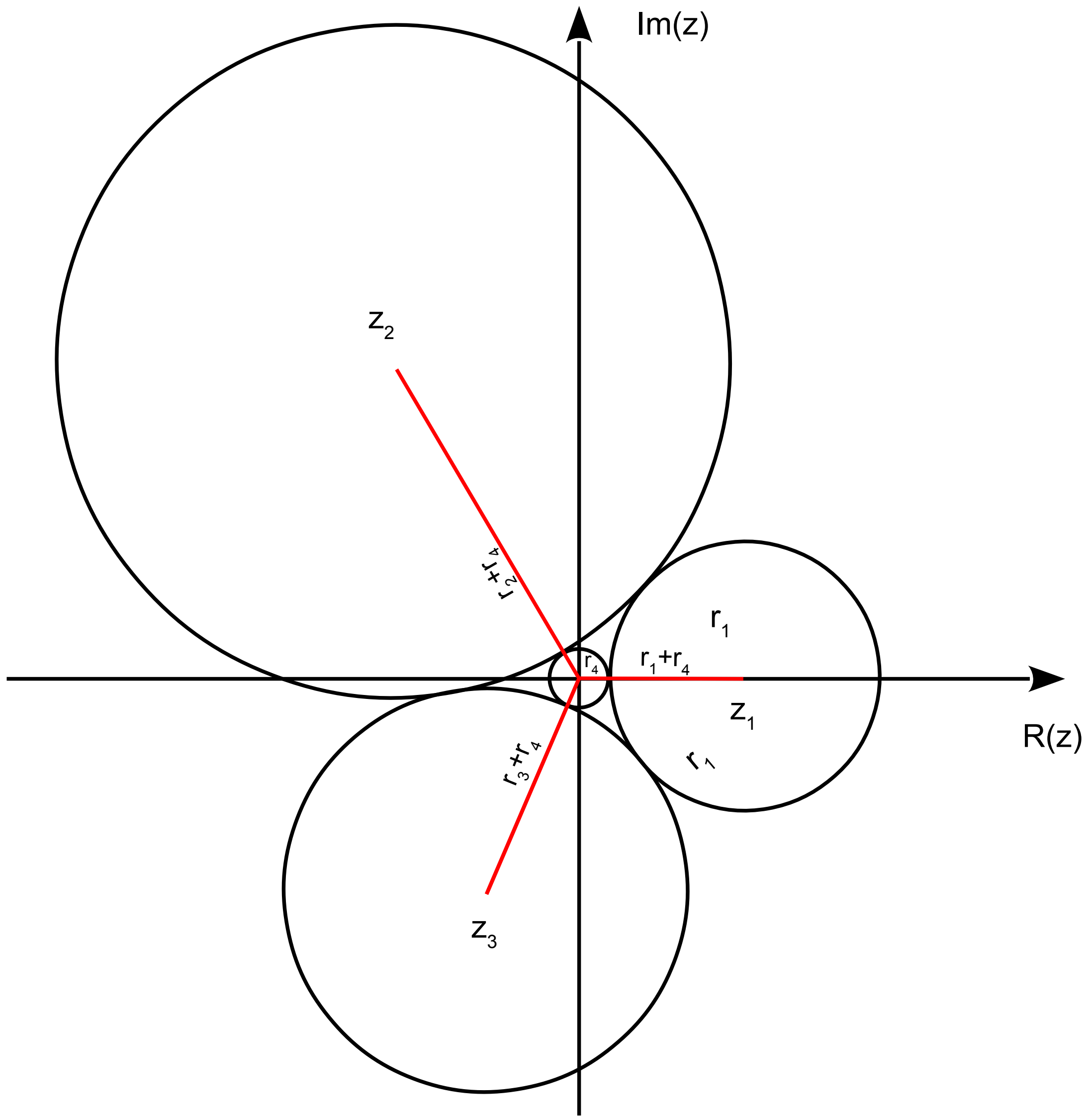
Man zeigt zunächst, dass die Gleichung **(6)** in einem speziellen Koordinatensystem richtig ist.

$$z_1 = r_1 + r_4$$

$$z_2 = (r_2 + r_4)e^{i\varphi_{12}}$$

$$z_3 = (r_3 + r_4)e^{-i\varphi_{31}} \qquad \text{mit} \qquad \varphi_{12} := \sphericalangle z_1 z_2 \quad , \quad \varphi_{31} := \sphericalangle z_3 z_4 z_1$$

$$z_4 = 0$$



In diesem speziellen Koordinatensystem lautet Gleichung **(6)** :

$$
\begin{aligned}
&\left[\frac{2}{r_1^2} - \frac{1}{r_1}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]\left(r_1+r_4\right) \\
&\quad + \left[\frac{2}{r_2^2} - \frac{1}{r_2}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]\left(r_2+r_4\right)e^{i\varphi_{12}} \\
&\qquad + \left[\frac{2}{r_3^2} - \frac{1}{r_3}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]\left(r_3+r_4\right)e^{-i\varphi_{31}} = 0
\end{aligned}
$$

Umformung ergibt :

$$
\begin{aligned}
&\left[\frac{2}{r_1^2} - \frac{1}{r_1^2} - \frac{1}{r_1}\left(\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]\left(r_1+r_4\right) \\
&\quad + \left[\frac{2}{r_2^2} - \frac{1}{r_2^2} - \frac{1}{r_2}\left(\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]\left(r_2+r_4\right)e^{i\varphi_{12}} \\
&\qquad + \left[\frac{2}{r_3^2} - \frac{1}{r_3^2} - \frac{1}{r_3}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_4}\right)\right]\left(r_3+r_4\right)e^{-i\varphi_{31}} = 0
\end{aligned}
$$

$$
\begin{aligned}
&\left[\frac{1}{r_1^2} - \frac{1}{r_1}\left(\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]\left(r_1+r_4\right) \\
&\quad + \left[\frac{1}{r_2^2} - \frac{1}{r_2}\left(\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]\left(r_2+r_4\right)e^{i\varphi_{12}} \\
&\qquad + \left[\frac{1}{r_3^2} - \frac{1}{r_3}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_4}\right)\right]\left(r_3+r_4\right)e^{-i\varphi_{31}} = 0
\end{aligned}
$$

Jetzt betrachtet man die Terme in den eckigen Klammern :

$$\frac{1}{r_1^2} - \frac{1}{r_1}\left(\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right) = \frac{1}{r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}-\frac{1}{r_3}-\frac{1}{r_4}\right)$$

$$\frac{1}{r_1^2} - \frac{1}{r_1}\left(\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right) = \frac{1}{r_1}\left(\frac{1}{r_1}+\frac{1}{r_4} - \frac{1}{r_2}-\frac{1}{r_4} - \frac{1}{r_3}-\frac{1}{r_4}\right)$$

$$\frac{1}{r_1^2} - \frac{1}{r_1}\left(\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right) = \frac{1}{r_1}\left(\frac{r_1+r_4}{r_1 r_4} - \frac{r_2+r_4}{r_2 r_4} - \frac{r_3+r_4}{r_3 r_4}\right)$$

$$\frac{1}{r_1^2} - \frac{1}{r_1}\left(\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right) = \frac{1}{r_1 r_4}\left(\frac{r_1+r_4}{r_1} - \frac{r_2+r_4}{r_2} - \frac{r_3+r_4}{r_3}\right)$$

$$\underline{\frac{1}{r_1^2} - \frac{1}{r_1}\left(\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right) = \frac{1}{r_1 r_4}\left(\frac{1}{\dfrac{r_1}{r_1+r_4}} - \frac{1}{\dfrac{r_2}{r_2+r_4}} - \frac{1}{\dfrac{r_3}{r_3+r_4}}\right)}$$

Analog durch zyklische Vertauschung :

$$\frac{1}{r_2^2} - \frac{1}{r_2}\left(\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_4}\right) = \frac{1}{r_2 r_4}\left(\frac{1}{\frac{r_2}{r_2+r_4}} - \frac{1}{\frac{r_3}{r_3+r_4}} - \frac{1}{\frac{r_1}{r_1+r_4}}\right)$$

$$\frac{1}{r_3^2} - \frac{1}{r_3}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_4}\right) = \frac{1}{r_3 r_4}\left(\frac{1}{\frac{r_3}{r_3+r_4}} - \frac{1}{\frac{r_1}{r_1+r_4}} - \frac{1}{\frac{r_2}{r_2+r_4}}\right)$$

$$\begin{aligned}
&\left[\frac{1}{r_1^2} - \frac{1}{r_1}\left(\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]\left(r_1+r_4\right) \\
&\quad + \left[\frac{1}{r_2^2} - \frac{1}{r_2}\left(\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_4}\right)\right]\left(r_2+r_4\right)e^{i\varphi_{12}} \\
&\qquad + \left[\frac{1}{r_3^2} - \frac{1}{r_3}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_4}\right)\right]\left(r_3+r_4\right)e^{-i\varphi_{31}} = 0
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{r_1 r_4}\left(\frac{1}{\frac{r_1}{r_1+r_4}} - \frac{1}{\frac{r_2}{r_2+r_4}} - \frac{1}{\frac{r_3}{r_3+r_4}}\right)\left(r_1+r_4\right) \\
&\quad + \frac{1}{r_2 r_4}\left(\frac{1}{\frac{r_2}{r_2+r_4}} - \frac{1}{\frac{r_3}{r_3+r_4}} - \frac{1}{\frac{r_1}{r_1+r_4}}\right)\left(r_2+r_4\right)e^{i\varphi_{12}} \\
&\qquad + \frac{1}{r_3 r_4}\left(\frac{1}{\frac{r_3}{r_3+r_4}} - \frac{1}{\frac{r_1}{r_1+r_4}} - \frac{1}{\frac{r_2}{r_2+r_4}}\right)\left(r_3+r_4\right)e^{-i\varphi_{31}} = 0
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{r_1}\left(\frac{1}{\frac{r_1}{r_1+r_4}} - \frac{1}{\frac{r_2}{r_2+r_4}} - \frac{1}{\frac{r_3}{r_3+r_4}}\right)\left(r_1+r_4\right) \\
&\quad + \frac{1}{r_2}\left(\frac{1}{\frac{r_2}{r_2+r_4}} - \frac{1}{\frac{r_3}{r_3+r_4}} - \frac{1}{\frac{r_1}{r_1+r_4}}\right)\left(r_2+r_4\right)e^{i\varphi_{12}} \\
&\qquad + \frac{1}{r_3}\left(\frac{1}{\frac{r_3}{r_3+r_4}} - \frac{1}{\frac{r_1}{r_1+r_4}} - \frac{1}{\frac{r_2}{r_2+r_4}}\right)\left(r_3+r_4\right)e^{-i\varphi_{31}} = 0
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\frac{r_1}{r_1+r_4}}\left(\frac{1}{\frac{r_1}{r_1+r_4}} - \frac{1}{\frac{r_2}{r_2+r_4}} - \frac{1}{\frac{r_3}{r_3+r_4}}\right) \\
&\quad + \frac{1}{\frac{r_2}{r_2+r_4}}\left(\frac{1}{\frac{r_2}{r_2+r_4}} - \frac{1}{\frac{r_3}{r_3+r_4}} - \frac{1}{\frac{r_1}{r_1+r_4}}\right)e^{i\varphi_{12}} \\
&\qquad + \frac{1}{\frac{r_3}{r_3+r_4}}\left(\frac{1}{\frac{r_3}{r_3+r_4}} - \frac{1}{\frac{r_1}{r_1+r_4}} - \frac{1}{\frac{r_2}{r_2+r_4}}\right)e^{-i\varphi_{31}} = 0
\end{aligned}$$

Setzt man vorübergehend

$$\alpha := \frac{1}{\frac{r_1}{r_1+r_4}} \qquad \beta := \frac{1}{\frac{r_2}{r_2+r_4}} \qquad \gamma := \frac{1}{\frac{r_3}{r_3+r_4}}$$

so folgt

$$\underbrace{\begin{aligned} &\frac{1}{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}\right) \\ &\quad + \frac{1}{\beta}\left(\frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\alpha}\right) e^{i\varphi_{12}} \\ &\qquad + \frac{1}{\gamma}\left(\frac{1}{\gamma} - \frac{1}{\alpha} - \frac{1}{\beta}\right) e^{-i\varphi_{31}} \end{aligned}}_{=:\ V} = 0$$

$$V = \frac{1}{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}\right) + \frac{1}{\beta}\left(\frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\alpha}\right) e^{i\varphi_{12}} + \frac{1}{\gamma}\left(\frac{1}{\gamma} - \frac{1}{\alpha} - \frac{1}{\beta}\right) e^{-i\varphi_{31}}$$

Man muss zeigen, dass $V = 0$, also $\mathrm{Re}(V) = 0$ und $\mathrm{Im}(V) = 0$.

$$\mathrm{Re}(V) = \frac{1}{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}\right) + \frac{1}{\beta}\left(\frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\alpha}\right) \cos\varphi_{12} + \frac{1}{\gamma}\left(\frac{1}{\gamma} - \frac{1}{\alpha} - \frac{1}{\beta}\right) \cos\varphi_{31}$$

Nach den Gleichungen **(1)** , **(3)** gilt

$$\cos\varphi_{12} = 1-2\frac{r_1 r_2}{(r_1+r_4)(r_2+r_4)} = 1-2\alpha\beta$$

$$\cos\varphi_{31} = 1-2\frac{r_3 r_1}{(r_3+r_4)(r_1+r_4)} = 1-2\gamma\alpha \qquad ,$$

und es folgt weiter

$$\mathrm{Re}(V) = \frac{1}{\alpha}\left(\frac{1}{\alpha}-\frac{1}{\beta}-\frac{1}{\gamma}\right) + \frac{1}{\beta}\left(\frac{1}{\beta}-\frac{1}{\gamma}-\frac{1}{\alpha}\right)(1-2\alpha\beta) + \frac{1}{\gamma}\left(\frac{1}{\gamma}-\frac{1}{\alpha}-\frac{1}{\beta}\right)(1-2\gamma\alpha)$$

$$\begin{aligned} \mathrm{Re}(V) = &\frac{1}{\alpha}\left(\frac{1}{\alpha}-\frac{1}{\beta}-\frac{1}{\gamma}\right) \\ &\quad + \frac{1}{\beta}\left(\frac{1}{\beta}-\frac{1}{\gamma}-\frac{1}{\alpha}\right)(1-2\alpha\beta) \\ &\qquad + \frac{1}{\gamma}\left(\frac{1}{\gamma}-\frac{1}{\alpha}-\frac{1}{\beta}\right)(1-2\gamma\alpha) \end{aligned}$$

$$\mathrm{Re}(V) = \frac{1}{\alpha}\left(\frac{1}{\alpha}-\frac{1}{\beta}-\frac{1}{\gamma}\right)$$
$$+ \frac{1}{\beta}\left(\frac{1}{\beta}-\frac{1}{\gamma}-\frac{1}{\alpha}\right) - 2\alpha\left(\frac{1}{\beta}-\frac{1}{\gamma}-\frac{1}{\alpha}\right)$$
$$+ \frac{1}{\gamma}\left(\frac{1}{\gamma}-\frac{1}{\alpha}-\frac{1}{\beta}\right) - 2\alpha\left(\frac{1}{\gamma}-\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$\mathrm{Re}(V) = \frac{1}{\alpha^2} - \frac{1}{\alpha\beta} - \frac{1}{\alpha\gamma}$$
$$+ \frac{1}{\beta^2} - \frac{1}{\beta\gamma} - \frac{1}{\beta\alpha} - 2\frac{\alpha}{\beta} + 2\frac{\alpha}{\gamma} + 2$$
$$+ \frac{1}{\gamma^2} - \frac{1}{\gamma\alpha} - \frac{1}{\gamma\beta} - 2\frac{\alpha}{\gamma} + 2 + 2\frac{\alpha}{\beta}$$

$$\mathrm{Re}(V) = \frac{1}{\alpha^2} - \frac{1}{\alpha\beta} - \frac{1}{\alpha\gamma}$$
$$+ \frac{1}{\beta^2} - \frac{1}{\beta\gamma} - \frac{1}{\beta\alpha} + 2$$
$$+ \frac{1}{\gamma^2} - \frac{1}{\gamma\alpha} - \frac{1}{\gamma\beta} + 2$$

$$\mathrm{Re}(V) = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - 2\left(\frac{1}{\alpha\beta} - \frac{1}{\alpha\gamma} - \frac{1}{\beta\gamma}\right) + 4$$

$$\mathrm{Re}(V) = 2\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right) - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + 2\left(\frac{1}{\alpha\beta} - \frac{1}{\alpha\gamma} - \frac{1}{\beta\gamma}\right)\right) + 4$$

$$\mathrm{Re}(V) = 2\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right) - \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)^2 + 4$$

Die Resubstitution

$$\alpha := \frac{1}{\frac{r_1}{r_1+r_4}} \qquad \beta := \frac{1}{\frac{r_2}{r_2+r_4}} \qquad \gamma := \frac{1}{\frac{r_3}{r_3+r_4}}$$

ergibt

$$\mathrm{Re}(V) = 2\left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)^2} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)^2} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)^2}\right) - \left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)}\right)^2 + 4 \quad .$$

Nach Gleichung **(4)** ist

$$2\left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)^2} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)^2} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)^2}\right) - \left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)}\right)^2 + 4 = 0 \quad ,$$

also

$$\mathrm{Re}(V) = 0 \quad .$$

Anstatt nun noch zu zeigen, dass $\operatorname{Im}(V) = 0$ ist, kann man auch zeigen, dass der Realteil der Drehung $Ve^{-i\varphi_{12}}$ gleich Null ist :

$$\operatorname{Re}\left(Ve^{-i\varphi_{12}}\right) = 0$$

$$V = \frac{1}{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}\right) + \frac{1}{\beta}\left(\frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\alpha}\right) e^{i\varphi_{12}} + \frac{1}{\gamma}\left(\frac{1}{\gamma} - \frac{1}{\alpha} - \frac{1}{\beta}\right) e^{-i\varphi_{31}}$$

$$V = \frac{1}{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}\right) + \frac{1}{\beta}\left(\frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\alpha}\right) e^{i\varphi_{12}} + \frac{1}{\gamma}\left(\frac{1}{\gamma} - \frac{1}{\alpha} - \frac{1}{\beta}\right) e^{i(\varphi_{12}+\varphi_{23})}$$

$$Ve^{-i\varphi_{12}} = \frac{1}{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}\right)e^{-i\varphi_{12}} + \frac{1}{\beta}\left(\frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\alpha}\right) + \frac{1}{\gamma}\left(\frac{1}{\gamma} - \frac{1}{\alpha} - \frac{1}{\beta}\right) e^{i\varphi_{23}}$$

$$Ve^{-i\varphi_{12}} = \frac{1}{\beta}\left(\frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\alpha}\right) + \frac{1}{\gamma}\left(\frac{1}{\gamma} - \frac{1}{\alpha} - \frac{1}{\beta}\right) e^{i\varphi_{23}} + \frac{1}{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}\right)e^{-i\varphi_{12}}$$

$$Ve^{-i\varphi_{12}} = \frac{1}{\beta}\left(\frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\alpha}\right) + \frac{1}{\gamma}\left(\frac{1}{\gamma} - \frac{1}{\alpha} - \frac{1}{\beta}\right) e^{i\varphi_{23}} + \frac{1}{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}\right)e^{i(\varphi_{23}+\varphi_{31})}$$

Die rechte Seite ist nur eine zyklische Vertauschung von $\operatorname{Re}(V)$.
Man muss nur α , β , γ durch β , γ , α und die Winkel φ_{12} , $-\varphi_{31}=\varphi_{12}+\varphi_{23}$ durch φ_{23} , $-\varphi_{12}=\varphi_{23}+\varphi_{31}$ ersetzen , und eine entsprechende Prozedur liefert $\operatorname{Re}\left(Ve^{-i\varphi_{12}}\right) = 0$.

Damit ist die Richtigkeit der Gleichung **(6)** gezeigt .

Zu Gleichung (7) :

Man zeigt nun die Gültigkeit der Gleichung

$$2\left(\frac{z_1^2}{r_1^2}+\frac{z_2^2}{r_2^2}+\frac{z_3^2}{r_3^2}+\frac{z_4^2}{r_4^2}\right) = \left(\frac{z_1}{r_1}+\frac{z_2}{r_2}+\frac{z_3}{r_3}+\frac{z_4}{r_4}\right)^2$$

in dem bereits verwendeten speziellen Koordinatensystem, wobei

$$\begin{aligned} z_1 &= r_1+r_4 \\ z_2 &= \left(r_2+r_4\right)e^{i\varphi_{12}} \\ z_3 &= \left(r_3+r_4\right)e^{-i\varphi_{31}} \qquad \text{mit} \quad \varphi_{12} := \sphericalangle z_1 z_2 \quad , \quad \varphi_{31} := \sphericalangle z_3 z_4 z_1 \\ z_4 &= 0 \end{aligned}$$

ist.

$$2\left(\frac{z_1^2}{r_1^2}+\frac{z_2^2}{r_2^2}+\frac{z_3^2}{r_3^2}+\frac{z_4^2}{r_4^2}\right) = \left(\frac{z_1}{r_1}+\frac{z_2}{r_2}+\frac{z_3}{r_3}+\frac{z_4}{r_4}\right)^2$$

$$2\left(\frac{\left(r_1+r_4\right)^2}{r_1^2} + \frac{\left(r_2+r_4\right)^2 e^{i2\varphi_{12}}}{r_2^2} + \frac{\left(r_3+r_4\right)^2 e^{-i2\varphi_{31}}}{r_3^2}\right) = \left(\frac{r_1+r_4}{r_1} + \frac{\left(r_2+r_4\right)e^{i\varphi_{12}}}{r_2} + \frac{\left(r_3+r_4\right)e^{-i\varphi_{31}}}{r_3}\right)^2$$

$$2\left(\frac{1}{\frac{r_1^2}{\left(r_1+r_4\right)^2}} + \frac{e^{i2\varphi_{12}}}{\frac{r_2^2}{\left(r_2+r_4\right)^2}} + \frac{e^{-i2\varphi_{31}}}{\frac{r_3^2}{\left(r_3+r_4\right)^2}}\right) = \left(\frac{1}{\frac{r_1}{r_1+r_4}} + \frac{e^{i\varphi_{12}}}{\frac{r_2}{r_2+r_4}} + \frac{e^{-i\varphi_{31}}}{\frac{r_3}{\left(r_3+r_4\right)}}\right)^2$$

Mit $\alpha := \dfrac{1}{\frac{r_1}{r_1+r_4}}$, $\beta := \dfrac{1}{\frac{r_2}{r_2+r_4}}$, $\gamma := \dfrac{1}{\frac{r_3}{r_3+r_4}}$

folgt :

$$2\left(\frac{1}{\alpha^2} + \frac{e^{i2\varphi_{12}}}{\beta^2} + \frac{e^{-i2\varphi_{31}}}{\gamma^2}\right) = \left(\frac{1}{\alpha} + \frac{e^{i\varphi_{12}}}{\beta} + \frac{e^{-i\varphi_{31}}}{\gamma}\right)^2$$

$$2\left(\frac{1}{\alpha^2} + \frac{e^{i2\varphi_{12}}}{\beta^2} + \frac{e^{-i2\varphi_{31}}}{\gamma^2}\right) = \frac{1}{\alpha^2} + \frac{e^{i2\varphi_{12}}}{\beta^2} + \frac{e^{-i2\varphi_{31}}}{\gamma^2} + 2\left(\frac{e^{i\varphi_{12}}}{\alpha\beta} + \frac{e^{-i\varphi_{31}}}{\alpha\gamma} + \frac{e^{i(\varphi_{12}-\varphi_{31})}}{\beta\gamma}\right)$$

$$\frac{1}{\alpha^2} + \frac{e^{i2\varphi_{12}}}{\beta^2} + \frac{e^{-i2\varphi_{31}}}{\gamma^2} = 2\left(\frac{e^{i\varphi_{12}}}{\alpha\beta} + \frac{e^{-i\varphi_{31}}}{\alpha\gamma} + \frac{e^{i(\varphi_{12}-\varphi_{31})}}{\beta\gamma}\right)$$

$$\underbrace{\frac{1}{\alpha^2} + \frac{e^{i2\varphi_{12}}}{\beta^2} + \frac{e^{-i2\varphi_{31}}}{\gamma^2} - 2\left(\frac{e^{i\varphi_{12}}}{\alpha\beta} + \frac{e^{-i\varphi_{31}}}{\alpha\gamma} + \frac{e^{i(\varphi_{12}-\varphi_{31})}}{\beta\gamma}\right)}_{=:\ W} = 0$$

$$W = \frac{1}{\alpha^2} + \frac{e^{i2\varphi_{12}}}{\beta^2} + \frac{e^{-i2\varphi_{31}}}{\gamma^2} - 2\left(\frac{e^{\hat{\imath}\varphi_{12}}}{\alpha\beta} + \frac{e^{-\hat{\imath}\varphi_{31}}}{\alpha\gamma} + \frac{e^{\hat{\imath}(\varphi_{12}-\varphi_{31})}}{\beta\gamma}\right) \tag{7.1}$$

$$\mathrm{Re}(W) = \frac{1}{\alpha^2} + \frac{\cos 2\varphi_{12}}{\beta^2} + \frac{\cos 2\varphi_{31}}{\gamma^2} - 2\frac{\cos\varphi_{12}}{\alpha\beta} - 2\frac{\cos\varphi_{31}}{\alpha\gamma} - 2\frac{\cos(\varphi_{12}-\varphi_{31})}{\beta\gamma}$$

Additionstheorem :

$$\begin{aligned}
\cos 2\varphi_{12} &= \cos^2\varphi_{12} - \sin^2\varphi_{12} \\
\cos 2\varphi_{12} &= \cos^2\varphi_{12} - 1 + \cos^2\varphi_{12} \\
\cos 2\varphi_{12} &= 2\cos^2\varphi_{12} - 1
\end{aligned}$$

Nach **(1)** , **(2)** ,**(3)** sind

$$\begin{aligned}
\cos\varphi_{12} &= 1 - 2\frac{r_1 r_2}{(r_1+r_4)(r_2+r_4)} = 1-2\alpha\gamma \\
\cos\varphi_{23} &= 1 - 2\frac{r_2 r_3}{(r_2+r_4)(r_3+r_4)} = 1-2\beta\gamma \\
\cos\varphi_{31} &= 1 - 2\frac{r_3 r_1}{(r_3+r_4)(r_1+r_4)} = 1-2\alpha\gamma \quad ,
\end{aligned}$$

also

$$\underline{\cos\varphi_{12} = 1 - 2\alpha\beta} \quad ,$$

und

$$\begin{aligned}
\cos 2\varphi_{12} &= 2\cos^2\varphi_{12} - 1 \\
\cos 2\varphi_{12} &= 2(1-2\alpha\beta)^2 - 1 \quad ,
\end{aligned}$$

und analog

$$\begin{aligned}
\cos\varphi_{31} &= 1 - 2\alpha\gamma \\
\cos 2\varphi_{31} &= 2(1-2\alpha\gamma)^2 - 1 \quad .
\end{aligned}$$

Additionstheorem und Symmetrieeigenschaften :

$$\left.\begin{aligned}
\cos(\varphi_{12}-\varphi_{31}) &= \cos\varphi_{12}\cos\varphi_{31} + \sin\varphi_{12}\sin\varphi_{31} \\
\cos\varphi_{23} &= \cos\left(2\pi-(\varphi_{12}+\varphi_{31})\right) \\
\cos\varphi_{23} &= \cos(\varphi_{12}+\varphi_{31}) \\
\cos\varphi_{23} &= \cos\varphi_{12}\cos\varphi_{31} - \sin\varphi_{12}\sin\varphi_{31}
\end{aligned}\right] +$$

$$\begin{aligned}
\cos(\varphi_{12}-\varphi_{31}) + \cos\varphi_{23} &= 2\cos\varphi_{12}\cos\varphi_{31} \\
\cos(\varphi_{12}-\varphi_{31}) &= 2\cos\varphi_{12}\cos\varphi_{31} - \cos\varphi_{23} \\
\cos(\varphi_{12}-\varphi_{31}) &= 2(1-2\alpha\beta)(1-2\alpha\gamma) - (1-2\beta\gamma)
\end{aligned}$$

$$\mathrm{Re}(W) = \frac{1}{\alpha^2} + \frac{\cos 2\varphi_{12}}{\beta^2} + \frac{\cos 2\varphi_{31}}{\gamma^2} - 2\frac{\cos\varphi_{12}}{\alpha\beta} - 2\frac{\cos\varphi_{31}}{\alpha\gamma} - 2\frac{\cos(\varphi_{12}-\varphi_{31})}{\beta\gamma}$$

$$\begin{aligned}\mathrm{Re}(W) = {} & \frac{1}{\alpha^2} + \frac{\left(2(1-2\alpha\beta)^2 - 1\right)}{\beta^2} + \frac{\left(2(1-2\alpha\gamma)^2 - 1\right)}{\gamma^2} \\ & - 2\frac{1 - 2\alpha\beta}{\alpha\beta} - 2\frac{1 - 2\alpha\gamma}{\alpha\gamma} - 2\frac{2(1-2\alpha\beta)(1-2\alpha\gamma) - (1-2\beta\gamma)}{\beta\gamma}\end{aligned}$$

$$\begin{aligned}\mathrm{Re}(W) = {} & \frac{1}{\alpha^2} + \frac{1-\alpha\beta+8\alpha^2\beta^2}{\beta^2} + \frac{1-8\alpha\gamma+8\alpha^2\gamma^2}{\gamma^2} \\ & - 2\frac{1 - 2\alpha\beta}{\alpha\beta} - 2\frac{1 - 2\alpha\gamma}{\alpha\gamma} - 2\frac{2(1-2\alpha\beta)(1-2\alpha\gamma) - (1-2\beta\gamma)}{\beta\gamma}\end{aligned}$$

$$\begin{aligned}\mathrm{Re}(W) = {} & \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{8\alpha}{\beta} + 8\alpha^2 + \frac{1}{\gamma^2} - \frac{8\alpha}{\gamma} + 8\alpha^2 \\ & - 2\frac{1 - 2\alpha\beta}{\alpha\beta} - 2\frac{1 - 2\alpha\gamma}{\alpha\gamma} - 2\frac{2(1-2\alpha\beta)(1-2\alpha\gamma) - (1-2\beta\gamma)}{\beta\gamma}\end{aligned}$$

$$\begin{aligned}\mathrm{Re}(W) = {} & \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{8\alpha}{\beta} + 8\alpha^2 + \frac{1}{\gamma^2} - \frac{8\alpha}{\gamma} + 8\alpha^2 \\ & - \frac{2}{\alpha\beta} + 4 - \frac{2}{\alpha\gamma} + 4 - 2\frac{2(1-2\alpha\beta)(1-2\alpha\gamma) - (1-2\beta\gamma)}{\beta\gamma}\end{aligned}$$

$$\begin{aligned}\mathrm{Re}(W) = {} & \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{8\alpha}{\beta} + 8\alpha^2 + \frac{1}{\gamma^2} - \frac{8\alpha}{\gamma} + 8\alpha^2 \\ & - \frac{2}{\alpha\beta} + 4 - \frac{2}{\alpha\gamma} + 4 - 2\frac{2(1-2\alpha\beta-2\alpha\gamma+4\alpha^2\beta\gamma) - 1 + 2\beta\gamma}{\beta\gamma}\end{aligned}$$

$$\begin{aligned}\mathrm{Re}(W) = {} & \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{8\alpha}{\beta} + 8\alpha^2 + \frac{1}{\gamma^2} - \frac{8\alpha}{\gamma} + 8\alpha^2 \\ & - \frac{2}{\alpha\beta} + 4 - \frac{2}{\alpha\gamma} + 4 - \frac{4(1-2\alpha\beta-2\alpha\gamma+4\alpha^2\beta\gamma) - 2 + 4\beta\gamma}{\beta\gamma}\end{aligned}$$

$$\begin{aligned}\mathrm{Re}(W) = {} & \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{8\alpha}{\beta} + 8\alpha^2 + \frac{1}{\gamma^2} - \frac{8\alpha}{\gamma} + 8\alpha^2 \\ & - \frac{2}{\alpha\beta} + 4 - \frac{2}{\alpha\gamma} + 4 - \frac{4-8\alpha\beta-8\alpha\gamma+16\alpha^2\beta\gamma - 2 + 4\beta\gamma}{\beta\gamma}\end{aligned}$$

$$\begin{aligned}\mathrm{Re}(W) = {} & \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{8\alpha}{\beta} + 8\alpha^2 + \frac{1}{\gamma^2} - \frac{8\alpha}{\gamma} + 8\alpha^2 \\ & - \frac{2}{\alpha\beta} + 4 - \frac{2}{\alpha\gamma} + 4 - \frac{2 - 8\alpha\beta - 8\alpha\gamma + 16\alpha^2\beta\gamma + 4\beta\gamma}{\beta\gamma}\end{aligned}$$

$$\begin{aligned}\mathrm{Re}(W) = {} & \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{8\alpha}{\beta} + 8\alpha^2 + \frac{1}{\gamma^2} - \frac{8\alpha}{\gamma} + 8\alpha^2 \\ & - \frac{2}{\alpha\beta} + 4 - \frac{2}{\alpha\gamma} + 4 - \frac{2}{\beta\gamma} + \frac{8\alpha}{\gamma} + \frac{8\alpha}{\beta} - 16\alpha^2 - 4\end{aligned}$$

$$\begin{aligned}\mathrm{Re}(W) = {} & \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} \\ & - \frac{2}{\alpha\beta} + 4 - \frac{2}{\alpha\gamma} - \frac{2}{\beta\gamma}\end{aligned}$$

$$\mathrm{Re}(W) = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\alpha\gamma} - \frac{2}{\beta\gamma} + 4$$

$$\operatorname{Re}(W) = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - 2\left(\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}\right) + 4$$

$$\operatorname{Re}(W) = 2\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right) - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + 2\left(\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}\right)\right) + 4$$

$$\operatorname{Re}(W) = 2\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right) - \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)^2 + 4$$

Die Resubstitution

$$\alpha := \frac{1}{\frac{r_1}{r_1+r_4}} \qquad \beta := \frac{1}{\frac{r_2}{r_2+r_4}} \qquad \gamma := \frac{1}{\frac{r_3}{r_3+r_4}}$$

ergibt

$$\operatorname{Re}(V) = 2\left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)^2} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)^2} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)^2}\right) - \left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)}\right)^2 + 4 \quad .$$

Nach Gleichung **(4)** ist

$$2\left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)^2} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)^2} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)^2}\right) - \left(\frac{1}{\left(\frac{r_1}{(r_1+r_4)}\right)} + \frac{1}{\left(\frac{r_2}{(r_2+r_4)}\right)} + \frac{1}{\left(\frac{r_3}{(r_3+r_4)}\right)}\right)^2 + 4 = 0 \quad ,$$

also

$$\operatorname{Re}(W) = 0 \quad .$$

Genau wie vorher, wird jetzt der Realteil von $\operatorname{Re}\left(W e^{-i\varphi_{12}}\right)$ betrachtet.
Man erhält eine zur Gleichung **(7.1)** zyklisch vertauschte Gleichung , entsprechend $\alpha, \beta, \gamma, \varphi_{12}, \varphi_{23}, \varphi_{31} \longrightarrow \beta, \gamma, \alpha, \varphi_{23}, \varphi_{31}, \varphi_{12}$.

Damit wird auch $\operatorname{Re}\left(W e^{-i\varphi_{12}}\right) = 0$, und die Richtigkeit der Gleichung **(7)** ist gezeigt.

q.e.d.