Minimale Summe der Abstandspotenzen

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Oktober 2019

Die vorliegende Arbeit bezieht sich auf das Skriptum „ **Arno Fehringer : Minimale Summe der Abstandskuben; Oktober 2019** “ und ist extrem knapp gehalten.

$$0 \le r \le h_a \quad , \quad 0 \le s \le h_b \quad , \quad 0 \le t \le h_c$$

$$|a|r \; + \; |b|s \; + \; |c|t \; + \; 2A_\Delta$$

$$t \; = \; \frac{1}{|c|}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right] \qquad , \qquad |a|r \; + \; |b|s \; \le \; 2A_\Delta$$

Summe der Abstandspotenzen

$$\boxed{f(r,s) \; = \; r^n \; + \; s^n \; + \; \frac{1}{|c|^n}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right]^n} \qquad , \qquad n \; \ge \; 2$$

Notwendige Bedingung für lokale Extrema

$$f_r(r,s) \; = \; n r^{n-1} \; - \; \frac{n|a|}{|c|^n}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right]^{n-1} \quad = \quad 0$$

$$f_s(r,s) \; = \; n s^{n-1} \; - \; \frac{n|b|}{|c|^n}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right]^{n-1} \quad = \quad 0$$

$$f_r(r,s) \; = \; r^{n-1} \; - \; \frac{|a|}{|c|^n}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right]^{n-1} \quad = \quad 0$$

$$f_s(r,s) \; = \; s^{n-1} \; - \; \frac{|b|}{|c|^n}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right]^{n-1} \quad = \quad 0$$

$$f_r(r,s) \; = \; r^{n-1} \; - \; \left(\frac{|a|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right]\right)^{n-1} \quad = \quad 0$$

$$f_s(r,s) \; = \; s^{n-1} \; - \; \left(\frac{|b|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right]\right)^{n-1} \quad = \quad 0$$

$$f_r(r,s) \; = \; \left(r \; - \; \frac{|a|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right]\right)\left(r^{n-2} \; + \; r^{n-3}\left\{\quad\right\} \; + \; \dots \; + \; \left\{\quad\right\}^{n-2}\right) \quad = \quad 0$$

$$\text{mit} \quad \left\{\quad\right\} \; = \; \frac{|a|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right]$$

$$f_s(r,s) \; = \; \left(s \; - \; \frac{|b|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right]\right)\left(s^{n-2} \; + \; s^{n-3}\left\{\quad\right\} \; + \; \dots \; + \; \left\{\quad\right\}^{n-2}\right) \quad = \quad 0$$

$$\text{mit} \quad \left\{\quad\right\} \; = \; \frac{|b|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}}\left[2A_\Delta \; - \; |a|r \; - \; |b|s\right]$$

1. Fall : $0<r<h_a \;,\; 0<s<h_b \;,\; 0<t<h_c \;,\; |a|r + |b|s < 2A_\Delta$

$$\left(r - \frac{|a|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}}\left[2A_\Delta - |a|r - |b|s\right]\right) = 0$$
$$\left(s - \frac{|b|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}}\left[2A_\Delta - |a|r - |b|s\right]\right) = 0$$

$$\left(1 + \frac{|a|^{\frac{n}{n-1}}}{|c|^{\frac{n}{n-1}}}\right) r + \frac{|a|^{\frac{1}{n-1}}|b|}{|c|^{\frac{n}{n-1}}} s = \frac{|a|^{\frac{1}{n-1}}2A_\Delta}{|c|^{\frac{n}{n-1}}}$$
$$\frac{|b|^{\frac{1}{n-1}}|a|}{|c|^{\frac{n}{n-1}}} r + \left(1 + \frac{|b|^{\frac{n}{n-1}}}{|c|^{\frac{n}{n-1}}}\right) s = \frac{|b|^{\frac{1}{n-1}}2A_\Delta}{|c|^{\frac{n}{n-1}}}$$

$$r = \frac{\frac{|a|^{\frac{1}{n-1}}2A_\Delta}{|c|^{\frac{n}{n-1}}}\left(1 + \frac{|b|^{\frac{n}{n-1}}}{|c|^{\frac{n}{n-1}}}\right) - \frac{|a|^{\frac{1}{n-1}}|b|}{|c|^{\frac{n}{n-1}}}\;\frac{|b|^{\frac{1}{n-1}}2A_\Delta}{|c|^{\frac{n}{n-1}}}}{\left(1 + \frac{|a|^{\frac{n}{n-1}}}{|c|^{\frac{n}{n-1}}}\right)\left(1 + \frac{|b|^{\frac{n}{n-1}}}{|c|^{\frac{n}{n-1}}}\right) - \frac{|a|^{\frac{1}{n-1}}|b|}{|c|^{\frac{n}{n-1}}}\;\frac{|b|^{\frac{1}{n-1}}|a|}{|c|^{\frac{n}{n-1}}}}$$

$$r = \frac{\frac{|a|^{\frac{1}{n-1}}2A_\Delta}{|c|^{\frac{n}{n-1}}}}{1 + \frac{|a|^{\frac{n}{n-1}}}{|c|^{\frac{n}{n-1}}} + \frac{|b|^{\frac{n}{n-1}}}{|c|^{\frac{n}{n-1}}}}$$

$$\boxed{r = \frac{|a|^{\frac{1}{n-1}}2A_\Delta}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}}}$$

$$\boxed{s = \frac{|b|^{\frac{1}{n-1}}2A_\Delta}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}}}$$

$$\boxed{t = \frac{|c|^{\frac{1}{n-1}}2A_\Delta}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}}}$$

$$\boxed{z = C + \frac{|a|}{2A_\Delta} b\ r - \frac{|b|}{2A_\Delta} a\ s}$$

$$z = C + \frac{|a|}{2A_\Delta}\ b\ \frac{|a|^{\frac{1}{n-1}} 2A_\Delta}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}} - \frac{|b|}{2A_\Delta}\ a\ \frac{|b|^{\frac{1}{n-1}} 2A_\Delta}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}}$$

$$z = C + b\ \frac{|a|^{\frac{n}{n-1}}}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}} - a\ \frac{|b|^{\frac{n}{n-1}}}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}}$$

$$z = \frac{C\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right) + b\ |a|^{\frac{n}{n-1}} - a\ |b|^{\frac{n}{n-1}}}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}}$$

$$z = \frac{C\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right) + (A-C)\ |a|^{\frac{n}{n-1}} - (C-B)\ |b|^{\frac{n}{n-1}}}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}}$$

$$\boxed{z = \frac{|a|^{\frac{n}{n-1}} A + |b|^{\frac{n}{n-1}} B + |c|^{\frac{n}{n-1}} C}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}}}$$

$$f(r,s) = r^n + s^n + \frac{1}{|c|^n}\left[2A_\Delta - |a|r - |b|s\right]^n$$

$$f(r,s) = r^n + s^n + t^n \quad , \qquad \frac{1}{|c|}\left[2A_\Delta - |a|r - |b|s\right]$$

$$f(r,s) = \frac{|a|^{\frac{n}{n-1}} 2^n A_\Delta^n}{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^n} + \frac{|b|^{\frac{n}{n-1}} 2^n A_\Delta^n}{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^n} + \frac{|c|^{\frac{n}{n-1}} 2^n A_\Delta^n}{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^n}$$

$$f(r,s) = \frac{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right) 2^n A_\Delta^n}{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^n}$$

$$\boxed{f(r,s) = \frac{2^n A_\Delta^n}{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^{n-1}}}$$

Zweite Ableitungen

$$f_{\underline{r}}(r,s) = n r^{n-1} - \frac{n|a|}{|c|^n}\left[2A_\Delta - |a|r - |b|s\right]^{n-1}$$
$$f_{\underline{s}}(r,s) = n s^{n-1} - \frac{n|b|}{|c|^n}\left[2A_\Delta - |a|r - |b|s\right]^{n-1}$$

$$f_{\underline{rr}}(r,s) = n(n-1)r^{n-2} + \frac{n(n-1)|a|^2}{|c|^n}\left[2A_\Delta - |a|r - |b|s\right]^{n-2}$$

$$f_{\underline{rr}}(r,s) = n(n-1)r^{n-2} + \frac{n(n-1)|a|^2}{|c|^n}\left[|c|t\right]^{n-2}$$

$$f_{\underline{rr}}(r,s) = n(n-1)\ r^{n-2} + \frac{n(n-1)|a|^2}{|c|^2}\ t^{n-2}$$

$$f_{\underline{rr}}(r,s) = n(n-1)\ \frac{|a|^{\frac{n-2}{n-1}}2^{n-2}A_\Delta^{n-2}}{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^{n-2}} + \frac{n(n-1)|a|^2}{|c|^2}\ \frac{|c|^{\frac{n-2}{n-1}}2^{n-2}A_\Delta^{n-2}}{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^{n-2}}$$

$$f_{\underline{rr}}(r,s) = \frac{n(n-1)|c|^2}{|c|^2}\ \frac{|a|^{\frac{n-2}{n-1}}2^{n-2}A_\Delta^{n-2}}{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^{n-2}} + \frac{n(n-1)|a|^2}{|c|^2}\ \frac{|c|^{\frac{n-2}{n-1}}2^{n-2}A_\Delta^{n-2}}{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^{n-2}}$$

$$\boxed{f_{\underline{rr}}(r,s) = \frac{n(n-1)2\,A_\Delta}{|c|^2\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^{n-2}}\ \left(|a|^{\frac{n-2}{n-1}}|c|^2 + |c|^{\frac{n-2}{n-1}}|a|^2\right)}$$

$$\boxed{f_{\underline{rr}}(r,s) \quad > \quad 0}$$

$$\boxed{f_{\underline{ss}}(r,s) = \frac{n(n-1)2^{n-2}A_\Delta^{n-2}}{|c|^2\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^{n-2}}\ \left(|b|^{\frac{n-2}{n-1}}|c|^2 + |c|^{\frac{n-2}{n-1}}|b|^2\right)}$$

$$f_{\underline{r}}(r,s) = n r^{n-1} - \frac{n|a|}{|c|^n}\left[2A_\Delta - |a|r - |b|s\right]^{n-1}$$

$$f_{\underline{rs}}(r,s) = \frac{n(n-1)|a||b|}{|c|^n}\left[2A_\Delta - |a|r - |b|s\right]^{n-2}$$

$$f_{\underline{rs}}(r,s) = \frac{n(n-1)|a||b|}{|c|^n}\left[|c|t\right]^{n-2}$$

$$f_{\underline{rs}}(r,s) = \frac{n(n-1)|a||b|}{|c|^2}\ t^{n-2}$$

$$f_{rs}(r,s) = \frac{n(n-1)|a||b|}{|c|^2} \frac{|c|^{\frac{n-2}{n-1}} 2^{n-2} A_\Delta^{\,n-2}}{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^{n-2}}$$

$$\boxed{f_{rs}(r,s) = \frac{n(n-1)2^{n-2}A_\Delta^{n-2}}{|c|^2\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^{n-2}} |c|^{\frac{n-2}{n-1}}|a||b|}$$

Setze $K := \dfrac{n(n-1)2^{n-2}A_\Delta^{n-2}}{|c|^2\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}\right)^{n-2}}$

$$f_{rr}(r,s) = K\left(|a|^{\frac{n-2}{n-1}}|c|^2 + |c|^{\frac{n-2}{n-1}}|a|^2\right)$$
$$f_{ss}(r,s) = K\left(|b|^{\frac{n-2}{n-1}}|c|^2 + |c|^{\frac{n-2}{n-1}}|b|^2\right)$$
$$f_{rs}(r,s) = K|c|^{\frac{n-2}{n-1}}|a||b|$$
$$f_{rs}^2(r,s) = K^2\,|c|^{\frac{2(n-2)}{n-1}}|a|^2|b|^2$$

$$\begin{aligned} &f_{rr}(r,s)f_{ss}(r,s) - f_{rs}^2(r,s) \\ &= K^2\left(|a|^{\frac{n-2}{n-1}}|c|^2 + |c|^{\frac{n-2}{n-1}}|a|^2\right)\left(|b|^{\frac{n-2}{n-1}}|c|^2 + |c|^{\frac{n-2}{n-1}}|b|^2\right) - K^2\,|c|^{\frac{2(n-2)}{n-1}}|a|^2|b|^2 \\ &= K^2\left(|a|^{\frac{n-2}{n-1}}|b|^{\frac{n-2}{n-1}}|c|^4 + |a|^{\frac{n-2}{n-1}}|c|^{\frac{n-2}{n-1}}|b|^2|c|^2 + |b|^{\frac{n-2}{n-1}}|c|^{\frac{n-2}{n-1}}|a|^2|c|^2\right) \end{aligned}$$

$$\boxed{f_{rr}(r,s)f_{ss}(r,s) - f_{rs}^2(r,s) = K^2\left(|a|^{\frac{n-2}{n-1}}|b|^{\frac{n-2}{n-1}}|c|^4 + |a|^{\frac{n-2}{n-1}}|c|^{\frac{n-2}{n-1}}|b|^2|c|^2 + |b|^{\frac{n-2}{n-1}}|c|^{\frac{n-2}{n-1}}|a|^2|c|^2\right)}$$

$$\boxed{f_{rr}(r,s)f_{ss}(r,s) - f_{rs}^2(r,s) \quad > \quad 0}$$

Damit ist der Punkt $\boxed{z = \dfrac{|a|^{\frac{n}{n-1}}A + |b|^{\frac{n}{n-1}}B + |c|^{\frac{n}{n-1}}C}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}}}$

ein Punkt, für den die **Summe der Abstandspotenzen lokal minimal** ist .

2. Fall : $0 \le r \le h_a \;,\; 0 \le s \le h_b \;,\; 0 \le t \le h_c \;,\; |a|r + |b|s \le 2A_\Delta$

$$\left(r^{n-2} + r^{n-3} \left\{\quad\right\} + \dots + \left\{\quad\right\}^{n-2} \right) = 0 \quad \text{mit} \quad \left\{\quad\right\} = \frac{|a|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}} \left[2A_\Delta - |a|r - |b|s \right]$$

$$\left(s^{n-2} + s^{n-3} \left\{\quad\right\} + \dots + \left\{\quad\right\}^{n-2} \right) = 0 \quad \text{mit} \quad \left\{\quad\right\} = \frac{|b|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}} \left[2A_\Delta - |a|r - |b|s \right]$$

$$r = 0 \quad \text{und} \quad \frac{|a|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}} \left[2A_\Delta - |a|r - |b|s \right] = 0 \quad \Rightarrow \quad t = 0 \quad \text{und} \quad s = h_b$$

$$s = 0 \quad \text{und} \quad \frac{|b|^{\frac{1}{n-1}}}{|c|^{\frac{n}{n-1}}} \left[2A_\Delta - |a|r - |b|s \right] = 0 \quad \Rightarrow \quad t = 0 \quad \text{und} \quad r = h_a$$

$$r = 0 \quad \text{und} \quad s = 0 \quad \Rightarrow \quad t = h_c$$

Die Punkte A **,** B **,** C **mit** $f(r,s,t) = h_a^n \;,\; h_b^n \;,\; h_c^n$ **kommen als lokale Extrema in Frage, aber :**

$$f(r,s) = \frac{2^n A_\Delta^n}{\left(|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}} \right)^{n-1}}$$

$$f(r,s) \le \frac{2^n A_\Delta^n}{\left(|a|^n + |b|^n + |c|^n \right)}$$

$$f(r,s) \le \frac{(2A_\Delta)^n}{\left(|a|^n + |b|^n + |c|^n \right)}$$

$$f(r,s) \le \frac{|a|^n}{\left(|a|^n + |b|^n + |c|^n \right)} \; h_a^{\,n}$$

$$f(r,s) < 1 \cdot h_a^{\,n}$$

$$\boxed{f(r,s) < h_a^{\,n}} \;,\; \boxed{f(r,s) < h_b^{\,n}} \;,\; \boxed{f(r,s) < h_c^{\,n}}$$

Damit ist der Punkt $\boxed{z = \frac{|a|^{\frac{n}{n-1}} A + |b|^{\frac{n}{n-1}} B + |c|^{\frac{n}{n-1}} C}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}}}$ **ein absolutes Minimum**

für die Summe der Abstandspotenzen .

Grenzfall

$$z_\infty = \lim_{n \to \infty} \frac{|a|^{\frac{n}{n-1}} A + |b|^{\frac{n}{n-1}} B + |c|^{\frac{n}{n-1}} C}{|a|^{\frac{n}{n-1}} + |b|^{\frac{n}{n-1}} + |c|^{\frac{n}{n-1}}}$$

$$\boxed{z_\infty = \frac{|a| A + |b| B + |c| C}{|a| + |b| + |c|}}$$

Inkreismittelpunkt des Dreiecks ΔABC .