Transversalen-Dreiecke

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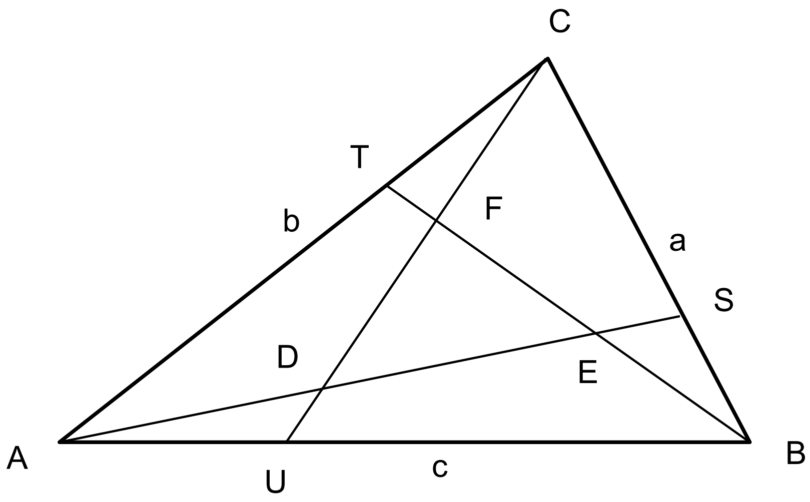
Quellen :

Fehringer, Arno : Geometrie in der komplexen Zahlenebene; Manuskript September 2019

Zu finden auf meiner Homepage

Fehringer, Arno : https://mathematikgarten.hpage.com/

Gegeben sei das Dreieck ΔABC in der komplexen Zahlenebene sowie drei Ecktransversalen, welche die Gegenseite jeweils im Verhältnis $t : (1-t)$, $t \in \mathbb{R}$ teilen.



$$S-B = t(C-B) = ta$$
$$T-C = t(A-C) = tb$$
$$U-A = t(B-A) = tc$$

Man berechne den Flächeninhalt des Dreiecks ΔDEF !

Berechnung des Punktes D :

$$AS : \quad (\bar{c}+t\bar{a})z - (c+ta)\bar{z} = (\bar{c}+t\bar{a})A - (c+ta)\bar{A}$$
$$CU : \quad (\bar{b}+t\bar{c})z - (b+tc)\bar{z} = (\bar{b}+t\bar{c})C - (b+tc)\bar{C}$$

$$D = \frac{((\bar{c}+t\bar{a})A - (c+ta)\bar{A})(b+tc) - (c+ta)((\bar{b}+t\bar{c})C - (b+tc)\bar{C})}{(\bar{c}+t\bar{a})(b+tc) - (c+ta)(\bar{b}+t\bar{c})}$$

Nenner N von D :

$$N = (\bar{c}+t\bar{a})(b+tc) - (c+ta)(\bar{b}+t\bar{c})$$

$$= \bar{c}b + \bar{c}ct + \bar{a}bt + \bar{a}ct^2 - c\bar{b} - c\bar{c}t - a\bar{b}t - a\bar{c}t^2$$

$$= \bar{c}b + \bar{a}bt + \bar{a}ct^2 - c\bar{b} - a\bar{b}t - a\bar{c}t^2$$

$$= \bar{c}b - c\bar{b} + \bar{a}bt - a\bar{b}t + \bar{a}ct^2 - a\bar{c}t^2$$

$$= \bar{c}b - c\bar{b} + (\bar{a}b - a\bar{b})t + (\bar{a}c - a\bar{c})t^2$$

$$= \bar{c}b - c\bar{b} - (\bar{b}a - b\bar{a})t + (\bar{a}c - a\bar{c})t^2$$

Wegen $A_{\Delta ABC} = \dfrac{\bar{c}b - c\bar{b}}{4} = \dfrac{\bar{b}a - b\bar{a}}{4} = \dfrac{\bar{a}c - a\bar{c}}{4}$ folgt :

$$\boxed{N = 4iA_{\Delta ABC}[1 - t + t^2]}$$

Zähler Z **von** D :

$$Z = ((\bar{c}+t\bar{a})A - (c+ta)\bar{A})(b+tc) - (c+ta)((\bar{b}+t\bar{c})C - (b+tc)\bar{C})$$

$$= (\bar{c}A - c\bar{A} + (\bar{a}A - a\bar{A})t)(b+tc) - (c+ta)(\bar{b}C - b\bar{C} + (\bar{c}C - c\bar{C})t)$$

$$= (\bar{c}A - c\bar{A})b - c(\bar{b}C - b\bar{C})$$

$$+ \left[\ (\bar{c}A - c\bar{A})c + (\bar{a}A - a\bar{A})b - (\bar{b}C - b\bar{C})a - (\bar{c}C - c\bar{C})c\ \right]t$$

$$+ \left[\ (\bar{a}A - a\bar{A})c - a(\bar{c}C - c\bar{C})\ \right]t^2$$

Berechnung des Absolutglieds von Z :

$$(\bar{c}A - c\bar{A})b - c(\bar{b}C - b\bar{C})$$

$$= \bar{c}bA - cb\bar{A} - c\bar{b}C + cb\bar{C} \qquad , \qquad A = C+b$$

$$= \bar{c}b(C+b) - cb(\bar{C}+\bar{b}) - c\bar{b}C + cb\bar{C}$$

$$= \bar{c}bC + \bar{c}bb - cb\bar{C} - cb\bar{b} - c\bar{b}C + cb\bar{C}$$

$$= \bar{c}bC + \bar{c}bb \cancel{- cb\bar{C}} - cb\bar{b} - c\bar{b}C \cancel{+ cb\bar{C}}$$

$$= \underline{\bar{c}bC} + \underline{\bar{c}bb} - \underline{cb\bar{b}} - \underline{c\bar{b}C}$$

$$= (\bar{c}b - c\bar{b})C + (\bar{c}b - c\bar{b})b$$

$$= (\bar{c}b - c\bar{b})[C + b]$$

$$= (\bar{c}b - c\bar{b})A \qquad , \qquad A_{\Delta ABC} = \frac{\bar{c}b - c\bar{b}}{4i}$$

$$\underline{= 4iA_{\Delta ABC}A}$$

Berechnung des Koeffizienten des linearen Terms von Z :

$$\left(\bar{c}A-c\bar{A}\right)c + \left(\bar{a}A-a\bar{A}\right)b - \left(\bar{b}C-b\bar{C}\right)a - \left(\bar{c}C-c\bar{C}\right)c$$

$$= \bar{c}cA - cc\bar{A} + \bar{a}bA - ab\bar{A} - \bar{b}aC + ba\bar{C} - \bar{c}cC + cc\bar{C}$$

$$A = C+b$$

$$= \bar{c}c\left(C+b\right) - cc\left(\bar{C}+\bar{b}\right) + \bar{a}b\left(C+b\right) - ab\left(\bar{C}+\bar{b}\right) - \bar{b}aC + ba\bar{C} - \bar{c}cC + cc\bar{C}$$

$$= \cancel{\bar{c}cC} + \bar{c}cb - \cancel{cc\bar{C}} - cc\bar{b} + \bar{a}bC + \bar{a}bb - \cancel{ab\bar{C}} - ab\bar{b} - \bar{b}aC + \cancel{ba\bar{C}} - \cancel{\bar{c}cC} + \cancel{cc\bar{C}}$$

$$= \underline{\bar{c}cb} - \underline{cc\bar{b}} + \underline{\bar{a}bC} + \underline{\bar{a}bb} - \underline{ab\bar{b}} - \underline{\bar{b}aC}$$

$$= \left(\bar{c}b - c\bar{b}\right)c - \left(\bar{b}a - b\bar{a}\right)C - \left(\bar{b}a - b\bar{a}\right)b$$

Wegen $A_{\Delta ABC} = \dfrac{\bar{c}b-c\bar{b}}{4i} = \dfrac{\bar{b}a-b\bar{a}}{4i} = \dfrac{\bar{a}c-a\bar{c}}{4i}$ folgt :

$$= 4iA_{\Delta ABC}\left(c - C - b\right)$$

$$= 4iA_{\Delta ABC}\left(c - \left(C + b\right)\right) \quad , \qquad C+b = A$$

$$= 4iA_{\Delta ABC}\left(c - A\right)$$

$$\underline{= -4iA_{\Delta ABC}\left(A - c\right)}$$

Berechnung des Koeffizienten des quadratischen Terms von Z :

$$\left(\bar{a}A-a\bar{A}\right)c - a\left(\bar{c}C-c\bar{C}\right)$$

$$= \bar{a}cA - ac\bar{A} - a\bar{c}C + ac\bar{C}$$

$$= \bar{a}c\left(C+b\right) - ac\left(\bar{C}+\bar{b}\right) - a\bar{c}C + ac\bar{C}$$

$$= \underline{\bar{a}cC} + \underline{\bar{a}cb} - \cancel{ac\bar{C}} - \underline{ac\bar{b}} - \underline{a\bar{c}C} + \cancel{ac\bar{C}}$$

$$= \left(\bar{a}c - a\bar{c}\right)C - \left(\bar{b}a - b\bar{a}\right)c \quad , \qquad A_{\Delta ABC} = \frac{\bar{c}b-c\bar{b}}{4i}$$

$$= 4iA_{\Delta ABC}\left(C - c\right) \quad , \qquad -c = b+a$$

$$= 4iA_{\Delta ABC}\left(C + b + a\right) \quad , \qquad C+b = A$$

$$\underline{= 4iA_{\Delta ABC}\left(A + a\right)}$$

Damit erhält man jetzt den Zähler Z **von** D **:**

$$Z = \left(\bar{c}A - c\bar{A}\right)b - c\left(\bar{b}C - b\bar{C}\right)$$
$$+ \left[\left(\bar{c}A - c\bar{A}\right)c + \left(\bar{a}A - a\bar{A}\right)b - \left(\bar{b}C - b\bar{C}\right)a - \left(\bar{c}C - c\bar{C}\right)c\right]t$$
$$+ \left[\left(\bar{a}A - a\bar{A}\right)c - a\left(\bar{c}C - c\bar{C}\right)\right]t^2$$

$$Z = 4iA_{\Delta ABC}A - 4iA_{\Delta ABC}(A-c)t + 4iA_{\Delta ABC}(A+a)t^2$$

$$\boxed{Z = 4iA_{\Delta ABC}\left[A - (A-c)t + (A+a)t^2\right]}$$

Damit folgt für den Transversalen-Schnittpunkt D **:**

$$D = \frac{Z}{N}$$

$$D = \frac{4iA_{\Delta ABC}\left[A - (A-c)t + (A+a)t^2\right]}{4iA_{\Delta ABC}\left[1 - t + t^2\right]}$$

$$\boxed{D = \frac{A - (A-c)t + (A+a)t^2}{1 - t + t^2}}$$

Aus **Symmetriegründen** sind dann die anderen Transversalen-Schnittpunkte

$$\boxed{E = \frac{B - (B-a)t + (B+b)t^2}{1 - t + t^2}}$$

$$\boxed{F = \frac{C - (C-b)t + (C+c)t^2}{1 - t + t^2}}$$

Berechnung der Seiten des Dreiecks ΔDEF **:**

$$d = F - E$$

$$d = \frac{(C-B) + \left[-(C-B)+b-a\right]t + \left[(C-B)+c-b\right]t^2}{1 - t + t^2}$$

$$d = \frac{a + \left[-a+b-a\right]t + \left[a+c-b\right]t^2}{1 - t + t^2} \quad , \quad a+c = -b$$

$$\boxed{d = \frac{a - \left[2a-b\right]t - 2bt^2}{1 - t + t^2}}$$

Durch **zyklische Vertauschung** erhält man die anderen Seiten :

$$\boxed{e = \frac{b - [2b-c]t - 2ct^2}{1 - t + t^2}}$$

$$\boxed{f = \frac{c - [2c-a]t - 2at^2}{1 - t + t^2}}$$

Der Flächeninhalt des Transversalen-Dreiecks ΔDEF **ist gegeben durch :**

$$A_{\Delta DEF} = \frac{1}{4i}\left(\bar{e}d - e\bar{d}\right)$$

$$A_{\Delta DEF} = \frac{1}{4i}\left(\frac{\bar{b}-[2\bar{b}-\bar{c}]t-2\bar{c}t^2}{1 - t + t^2}\;\frac{a-[2a-b]t-2at^2}{1 - t + t^2} - \frac{b-[2b-c]t-2ct^2}{1 - t + t^2}\;\frac{\bar{a}-[2\bar{a}-\bar{b}]t-2\bar{b}t^2}{1 - t + t^2}\right)$$

$$A_{\Delta DEF} = \frac{1}{4i\left(1-t+t^2\right)^2}\left(\left(\bar{b}-[2\bar{b}-\bar{c}]t-2\bar{c}t^2\right)\left(a-[2a-b]t-2at^2\right) - \left(b-[2b-c]t-2ct^2\right)\left(\bar{a}-[2\bar{a}-\bar{b}]t-2\bar{b}t^2\right)\right)$$

Nebenrechnung :

$$\left(\bar{b}-[2\bar{b}-\bar{c}]t-2\bar{c}t^2\right)\left(a-[2a-b]t-2at^2\right) - \left(b-[2b-c]t-2ct^2\right)\left(\bar{a}-[2\bar{a}-\bar{b}]t-2\bar{b}t^2\right)$$

$$= \bar{b}a - [2\bar{b}-\bar{c}]at - 2\bar{c}at^2$$

$$- \bar{b}[2a-b]t + [2\bar{b}-\bar{c}][2a-b]t^2 + 2\bar{c}[2a-b]t^3$$

$$\cancel{-2\bar{b}bt^2} + [2\bar{b}-\bar{c}]2bt^3 + 4\bar{c}bt^4$$

$$- \left[\begin{array}{l} b\bar{a} - [2b-c]\bar{a}t - 2c\bar{a}t^2 \\ \quad - b[2\bar{a}-\bar{b}]t + [2b-c][2\bar{a}-\bar{b}]t^2 + 2c[2\bar{a}-\bar{b}]t^3 \\ \quad\quad \cancel{-2b\bar{b}t^2} + [2b-c]2\bar{b}t^3 + 4c\bar{b}t^4 \end{array}\right]$$

$$
\begin{aligned}
= \;& \bar b a - [2\bar b - \bar c] a t - 2\bar c a t^2 \\
& - \bar b [2a - b] t + [2\bar b - \bar c][2a - b] t^2 + 2\bar c [2a - b] t^3 \\
& \qquad + [2\bar b - \bar c] 2 b t^3 + 4\bar c b t^4
\end{aligned}
$$

$$
\begin{aligned}
= \;& \bar b a - 2\bar b a t + \bar c a t - 2\bar c a t^2 \\
& - 2\bar b a t \cancel{+ \bar b b t} + 4\bar b a t^2 - \cancel{2\bar b b t^2} - 2a\bar c t^2 + \bar c b t^2 + 4\bar c a t^3 - 2\bar c b t^3 \\
& \qquad \cancel{+ 4\bar b b t^3} - 2\bar c b t^3 + 4\bar c b t^4 \\
& - \left[\begin{aligned} & b\bar a - 2b\bar a t + c\bar a t - 2c\bar a t^2 \\ & - 2b\bar a t \cancel{+ b\bar b t} + 4b\bar a t^2 - \cancel{2b\bar b t^2} - 2\bar a c t^2 + c\bar b t^2 + 4c\bar a t^3 - 2c\bar b t^3 \\ & \qquad \cancel{+ 4b\bar b t^3} - 2c\bar b t^3 + 4c\bar b t^4 \end{aligned} \right]
\end{aligned}
$$

$$
\begin{aligned}
= \;& \bar b a - b\bar a \\
& -2(\bar b a - b\bar a) t - (\bar a c - a\bar c) t - 2(\bar b a - b\bar a) t \\
& + 2(\bar a c - a\bar c) t^2 + 4(\bar b a - b\bar a) t^2 + 2(\bar a c - a\bar c) t^2 + (\bar c b - c\bar b) t^2 \\
& -4(\bar a c - a\bar c) t^3 - 2(\bar c b - c\bar b) t^3 - 2(\bar c b - c\bar b) t^3 \\
& +4(\bar c b - c\bar b) t^4
\end{aligned}
$$

$$
\begin{aligned}
= \;& 4iA_{\Delta ABC} \\
& -2(4iA_{\Delta ABC}) t - (4iA_{\Delta ABC}) t - 2(4iA_{\Delta ABC}) t \\
& + 2(4iA_{\Delta ABC}) t^2 + 4(4iA_{\Delta ABC}) t^2 + 2(4iA_{\Delta ABC}) t^2 + (4iA_{\Delta ABC}) t^2 \\
& -4(4iA_{\Delta ABC}) t^3 - 2(4iA_{\Delta ABC}) t^3 - 2(4iA_{\Delta ABC}) t^3 \\
& +4(4iA_{\Delta ABC}) t^4
\end{aligned}
$$

$$
\underline{= \; 4iA_{\Delta ABC}\left[1 - 5t + 9t^2 - 8t^3 + 4t^4\right]}
$$

Damit erhält man den Flächeninhalt des Dreiecks ΔDEF :

$$A_{\Delta DEF} = \frac{1}{4i\left(1-t+t^2\right)^2}\left(\left(\bar{b}-[2\bar{b}-\bar{c}]t-2\bar{c}t^2\right)\left(a-[2a-b]t-2at^2\right) - \left(b-[2b-c]t-2ct^2\right)\left(\bar{a}-[2\bar{a}-\bar{b}]t-2\bar{b}t^2\right)\right)$$

$$A_{\Delta DEF} = \frac{1}{4i\left(1-t+t^2\right)^2}\, 4iA_{\Delta ABC}\left[1 - 5t + 9t^2 - 8t^3 + 4t^4\right]$$

$$\boxed{A_{\Delta DEF} = \frac{\left[1 - 5t + 9t^2 - 8t^3 + 4t^4\right]}{\left[1 - t + t^2\right]^2}\, A_{\Delta ABC}}$$

und

$$\boxed{\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \frac{\left[1 - 5t + 9t^2 - 8t^3 + 4t^4\right]}{\left[1 - t + t^2\right]^2}}$$

Der Zähler kann mittels Polynomdivision faktorisiert werden, da dieser für $t = \frac{1}{2}$ eine Nullstelle hat (Das Dreieck ΔDEF schrumpft zum Schwerpunkt zusammen) :

$$\left[4t^4 - 8t^3 + 9t^2 - 5t + 1\right] : \left[t - \frac{1}{2}\right] = 4t^3 - 6t^2 + 6t - 2$$

$$\left[4t^3 - 6t^2 + 6t - 2\right] : \left[t - \frac{1}{2}\right] = 4t^2 - 4t + 4 = 4\left[t^2 - t + 1\right]$$

$$\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \frac{\left[t - \frac{1}{2}\right]^2 4\left[t^2 - t + 1\right]}{\left[1 - t + t^2\right]^2}$$

$$\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \frac{4\left[t - \frac{1}{2}\right]^2}{\left[t^2 - t + 1\right]}$$

$$\boxed{\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \frac{4\left[t^2 - t + \frac{1}{4}\right]}{\left[t^2 - t + 1\right]}}$$

Für $t \to \pm\infty$ erhält man ein Dreieck derart, dass das Dreieck ΔABC dessen Seitenmitten-Dreieck ist, und dieses Dreieck den 4-fachen Flächeninhalt hat :

$$\boxed{\lim_{t \to \pm\infty} \frac{4\left[t^2 - t + \frac{1}{4}\right]}{\left[t^2 - t + 1\right]} = 4}$$

$$v(t) := \frac{4\left[t - \frac{1}{2}\right]^2}{\left[t^2 - t + 1\right]}$$

$$v(t) = \frac{4\left[t - \frac{1}{2}\right]^2}{\left[t - \frac{1}{2}\right]^2 + \frac{3}{4}}$$ **symmetrisch** bezüglich $t = \frac{1}{2}$