Allgemeine Transversalen-Dreiecke

Arno Fehringer

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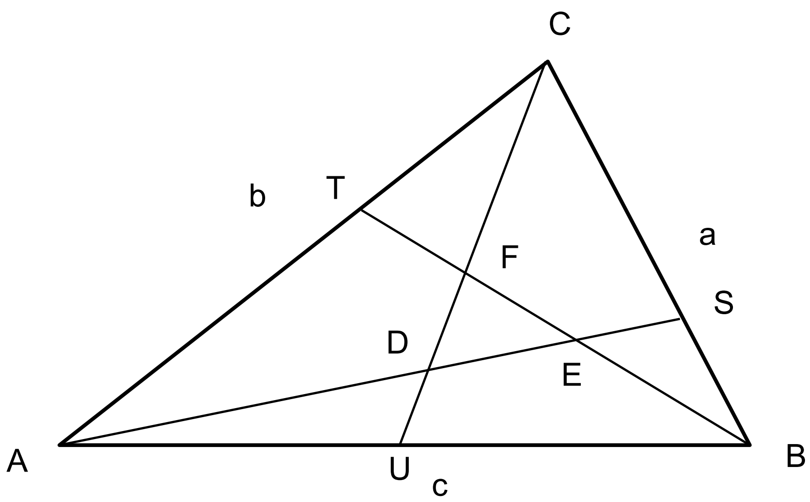
Quellen :

Fehringer, Arno : Geometrie in der komplexen Zahlenebene; Manuskript September 2019

Zu finden auf meiner Homepage

Fehringer, Arno : https://mathematikgarten.hpage.com/

Gegeben sei das Dreieck ΔABC in der komplexen Zahlenebene sowie drei Ecktransversalen, welche die Gegenseite jeweils im Verhältnis $s : (1-s)$, $t : (1-t)$, $u : (1-u)$ mit $s, t, u \in \mathbb{R}$ teilen.



$$\begin{aligned} S-B &= s(C-B) = sa \\ T-C &= t(A-C) = tb \\ U-A &= u(B-A) = uc \end{aligned}$$

Man berechne den Flächeninhalt des Dreiecks ΔDEF !

Berechnung des Punktes D :

$$\begin{aligned} AS &: \quad (\bar{c}+\bar{a}s)z - (c+as)\bar{z} = (\bar{c}+\bar{a}s)A - (c+as)\bar{A} \\ CU &: \quad (\bar{b}+\bar{c}u)z - (b+cu)\bar{z} = (\bar{b}+\bar{c}u)C - (b+cu)\bar{C} \end{aligned}$$

$$D = \frac{\left((\bar{c}+\bar{a}s)A - (c+a)\bar{A}s\right)(b+cu) - (c+as)\left((\bar{b}+\bar{c}u)C - (b+cu)\bar{C}\right)}{(\bar{c}+\bar{a}s)(b+cu) - (c+as)(\bar{b}+\bar{c}u)}$$

Nenner N von D :

$$\begin{aligned} N &= (\bar{c}+\bar{a}s)(b+cu) - (c+as)(\bar{b}+\bar{c}u) \\ &= \bar{c}b + \cancel{\bar{c}cu} + \bar{a}bs + \bar{a}csu - c\bar{b} - \cancel{c\bar{c}u} - a\bar{b}s - a\bar{c}su \\ &= \bar{c}b - c\bar{b} - a\bar{b}s + \bar{a}bs + \bar{a}csu - a\bar{c}su \\ &= \bar{c}b - c\bar{b} - (\bar{b}a - b\bar{a})s + (\bar{a}c - a\bar{c})su \end{aligned}$$

Wegen $A_{\Delta ABC} = \dfrac{\bar{c}b-c\bar{b}}{4} = \dfrac{\bar{b}a-b\bar{a}}{4} = \dfrac{\bar{a}c-a\bar{c}}{4}$ folgt :

$$\boxed{N = 4iA_{\Delta ABC}\left[1 - s + su\right]}$$

Zähler Z von D :

$$Z = \left(\left(\bar{c}+\bar{a}s\right)A - \left(c+as\right)\bar{A}\right)\left(b+cu\right) - \left(c+as\right)\left(\left(\bar{b}+\bar{c}u\right)C - \left(b+cu\right)\bar{C}\right)$$

$$= \left(\bar{c}A-c\bar{A} + \left(\bar{a}A-a\bar{A}\right)s\right)\left(b+cu\right) - \left(c+as\right)\left(\bar{b}C-b\bar{C} + \left(\bar{c}C-c\bar{C}\right)u\right)$$

$$= \left(\bar{c}A-c\bar{A}\right)b - c\left(\bar{b}C-b\bar{C}\right)$$

$$+ \left[\ \left(\bar{a}A-a\bar{A}\right)b - \left(\bar{b}C-b\bar{C}\right)a\ \right]s + \left[\ \left(\bar{c}A-c\bar{A}\right)c - c\left(\bar{c}C-c\bar{C}\right)\ \right]u$$

$$+ \left[\ \left(\bar{a}A-a\bar{A}\right)c - a\left(\bar{c}C-c\bar{C}\right)\ \right]su$$

Berechnung des Absolutglieds von Z **:**

$$\left(\bar{c}A-c\bar{A}\right)b - c\left(\bar{b}C-b\bar{C}\right)$$

$$= \bar{c}bA - cb\bar{A} - c\bar{b}C + cb\bar{C} \qquad , \qquad A = C+b$$

$$= \bar{c}b\left(C+b\right) - cb\left(\bar{C}+\bar{b}\right) - c\bar{b}C + cb\bar{C}$$

$$= \bar{c}bC + \bar{c}bb - cb\bar{C} - cb\bar{b} - c\bar{b}C + cb\bar{C}$$

$$= \bar{c}bC + \bar{c}bb \cancel{- cb\bar{C}} - cb\bar{b} - c\bar{b}C \cancel{+ cb\bar{C}}$$

$$= \underline{\bar{c}bC} + \underline{\bar{c}bb} - \underline{cb\bar{b}} - \underline{c\bar{b}C}$$

$$= \left(\bar{c}b - c\bar{b}\right)C + \left(\bar{c}b - c\bar{b}\right)b$$

$$= \left(\bar{c}b - c\bar{b}\right)\left[C + b\right]$$

$$= \left(\bar{c}b - c\bar{b}\right)A \qquad , \qquad A_{\Delta ABC} = \frac{\bar{c}b-c\bar{b}}{4i}$$

$$= \underline{4iA_{\Delta ABC}\ A}$$

Berechnung des Koeffizienten des linearen Terms von Z **bezüglich** s **:**

$$\left(\bar{a}A-a\bar{A}\right)b - \left(\bar{b}C-b\bar{C}\right)a$$

$$= \bar{a}bA - ab\bar{A} - \bar{b}aC + ba\bar{C}$$

$$= \bar{a}b\left(C+b\right) - ab\left(\bar{C}+\bar{b}\right) - \bar{b}aC + ba\bar{C}$$

$$= \bar{a}bC + \bar{a}bb \cancel{- ab\bar{C}} - ab\bar{b} - \bar{b}aC \cancel{+ ba\bar{C}}$$

$$= \bar{a}bC + \bar{a}bb - ab\bar{b} - \bar{b}aC$$

$$= -(\bar{b}a - b\bar{a})C - (\bar{b}a - b\bar{a})b$$
$$= -(\bar{b}a - b\bar{a})(C + b)$$
$$= -(\bar{b}a - b\bar{a})A$$

$$\underline{= -4iA_{\Delta ABC}\;A}$$

Berechnung des Koeffizienten des linearen Terms von Z bezüglich u :

$$(\bar{c}A - c\overline{A})c - c(\bar{c}C - c\overline{C})$$

$$= \bar{c}cA - cc\overline{A} - \bar{c}cC + cc\overline{C}$$
$$= \bar{c}c(C+b) - cc(\overline{C}+\bar{b}) - \bar{c}cC + cc\overline{C}$$
$$= \cancel{\bar{c}cC} + \bar{c}cb - \cancel{cc\overline{C}} - cc\bar{b} - \cancel{\bar{c}cC} + \cancel{cc\overline{C}}$$

$$= \bar{c}cb - cc\bar{b}$$

$$= (\bar{c}b - c\bar{b})c$$
$$\underline{= -4iA_{\Delta ABC}\;c}$$

Berechnung des Koeffizienten des quadratischen Terms von Z :

$$(\bar{a}A - a\overline{A})c - a(\bar{c}C - c\overline{C})$$

$$= \bar{a}cA - ac\overline{A} - a\bar{c}C + ac\overline{C}$$

$$= \bar{a}c(C+b) - ac(\overline{C}+\bar{b}) - a\bar{c}C + ac\overline{C}$$

$$= \underline{\bar{a}cC} + \underline{\bar{a}cb} - \cancel{ac\overline{C}} - \underline{ac\bar{b}} - \underline{a\bar{c}C} + \cancel{ac\overline{C}}$$

$$= (\bar{a}c - a\bar{c})C - (\bar{b}a - b\bar{a})c \qquad , \qquad A_{\Delta ABC} = \frac{\bar{c}b - c\bar{b}}{4i}$$

$$= 4iA_{\Delta ABC}(C - c) \qquad , \qquad -c = b+a$$

$$= 4iA_{\Delta ABC}(C + b + a) \qquad , \qquad C+b = A$$

$$\underline{= 4iA_{\Delta ABC}(A + a)}$$

Damit erhält man jetzt den Zähler Z von D :

$$Z = (\bar{c}A - c\overline{A})b - c(\bar{b}C - b\overline{C})$$
$$+ \left[\, (\bar{a}A - a\overline{A})b - (\bar{b}C - b\overline{C})a \,\right]s + \left[\, (\bar{c}A - c\overline{A})c - c(\bar{c}C - c\overline{C}) \,\right]u$$
$$+ \left[\, (\bar{a}A - a\overline{A})c - a(\bar{c}C - c\overline{C}) \,\right]su$$

$$Z = 4iA_{\Delta ABC}\,A - 4iA_{\Delta ABC}\,As + 4iA_{\Delta ABC}\,cu + 4iA_{\Delta ABC}(A+a)su$$

$$\boxed{Z = 4iA_{\Delta ABC}\left[\,A - As + cu + (A+a)su\,\right]}$$

Damit folgt für den Transversalen-Schnittpunkt D **:**

$$D = \frac{Z}{N}$$

$$D = \frac{4iA_{\Delta ABC}\left[\,A - As + cu + (A+a)su\,\right]}{4iA_{\Delta ABC}\left[1 - s + su\right]}$$

$$\boxed{D = \frac{\left[\,A - As + cu + (A+a)su\,\right]}{\left[1 - s + su\right]}}$$

Aus **Symmetriegründen** sind dann die anderen Transversalen-Schnittpunkte

$$\boxed{E = \frac{\left[\,B - Bt + as + (B+b)ts\,\right]}{\left[1 - t + ts\right]}}$$

$$\boxed{F = \frac{\left[\,C - Cu + bt + (C+c)ut\,\right]}{\left[1 - u + ut\right]}}$$

Andere Darstellung der Transversalen-Schnittpunkte :

$$D = \frac{\left[A - As + cu + (A+a)su\right]}{\left[1 - s + su\right]}$$

$$D = \frac{\left[A - As + cu + Asu + asu\right]}{\left[1 - s + su\right]}$$

$$D = \frac{A\left[1 - s + su\right] + \left[c + as\right]u}{\left[1 - s + su\right]}$$

$$D = A + \frac{u}{\left[1-s+su\right]}\left[c+as\right]$$

$$\boxed{D = A + \frac{u}{\left[1-s\right]\left[1-u\right] + u}\left[c+as\right]}$$

Aus **Symmetriegründen** sind dann die anderen Transversalen-Schnittpunkte

$$\boxed{E = B + \frac{s}{\left[1-t\right]\left[1-s\right] + s}\left[a+bt\right]}$$

$$\boxed{F = C + \frac{t}{\left[1-u\right]\left[1-t\right] + t}\left[b+cu\right]}$$

Berechnung der Seiten des Dreiecks ΔDEF :

$$d = F-E \quad , \quad e = D-F \quad , \quad f = E-D$$

$$d = C + \frac{t}{[1-u][1-t] + t}\,[b+cu] - \left(B + \frac{s}{[1-t][1-s] + s}\,[a+bt]\right)$$

$$d = C - B + \frac{t}{[1-u][1-t] + t}\,[b+cu] - \frac{s}{[1-t][1-s] + s}\,[a+bt]$$

$$\boxed{d = a + \frac{t}{[1-u][1-t] + t}\,[b+cu] - \frac{s}{[1-t][1-s] + s}\,[a+bt]}$$

$$\boxed{e = b + \frac{u}{[1-s][1-u] + u}\,[c+as] - \frac{t}{[1-u][1-t] + t}\,[b+cu]}$$

$$\boxed{f = c + \frac{s}{[1-t][1-s] + s}\,[a+bt] - \frac{u}{[1-s][1-u] + u}\,[c+as]}$$

Der Flächeninhalt des Dreiecks ΔDEF ist gegeben durch :

$$A_{\Delta DEF} = \frac{\bar{e}d - e\bar{d}}{4i}$$

$$A_{\Delta DEF} = \frac{1}{4i}\left(\bar{e}d - e\bar{d}\right)$$

$$A_{\Delta DEF} = \frac{1}{4i}\; e\times d \qquad \text{mit} \qquad e\times d := \bar{e}d - e\bar{d}$$

Setzt man vorübergehend

$$\alpha := \frac{s}{[1-t][1-s] + s} \qquad \beta := \frac{t}{[1-u][1-t] + t} \qquad \gamma := \frac{u}{[1-s][1-u] + u} \quad ,$$

so folgt :

$$d = a + \frac{t}{[1-u][1-t] + t}\,[b+cu] - \frac{s}{[1-t][1-s] + s}\,[a+bt]$$

$$e = b + \frac{u}{[1-s][1-u] + u}\,[c+as] - \frac{t}{[1-u][1-t] + t}\,[b+cu]$$

$$d = a + \beta\,[b+cu] - \alpha\,[a+bt]$$

$$e = b + \gamma[c+as] - \beta[b+cu]$$

$$d = a + \beta b + \beta uc - \alpha a - \alpha tb$$

$$e = b + \gamma c + \gamma sa - \beta b - \beta uc$$

$$e\times d = (b + \gamma c + \gamma sa - \beta b + \beta uc) \times (a + \beta b + \beta uc - \alpha a - \alpha tb)$$

$$\begin{aligned} e\times d = {} & (b + \gamma c + \gamma sa - \beta b - \beta uc) \times a \\ & + (b + \gamma c + \gamma sa - \beta b - \beta uc) \times \beta b \\ & + (b + \gamma c + \gamma sa - \beta b - \beta uc) \times \beta uc \\ & + (b + \gamma c + \gamma sa - \beta b - \beta uc) \times -\alpha a \\ & + (b + \gamma c + \gamma sa - \beta b - \beta uc) \times -\alpha tb \end{aligned}$$

$$\begin{aligned} e\times d = {} & + b\times a + \gamma c\times a + \gamma sa\times a - \beta b\times a - \beta uc\times a \\ & + \beta b\times b + \gamma\beta c\times b + \gamma\beta sa\times b - \beta^2 b\times b - \beta^2 uc\times b \\ & + \beta ub\times c + \gamma\beta uc\times c + \gamma\beta usa\times c - \beta^2 ub\times c - \beta^2 u^2 c\times c \\ & -(\alpha b\times a + \gamma\alpha c\times a + \gamma\alpha sa\times a - \beta\alpha b\times a - \beta\alpha uc\times a) \\ & -(\alpha tb\times b + \gamma\alpha tc\times b + \gamma\alpha tsa\times b - \beta\alpha tb\times b - \beta\alpha tuc\times b) \end{aligned}$$

$$\begin{aligned} e\times d = {} & + b\times a + \gamma c\times a + \gamma sa\times a - \beta b\times a - \beta uc\times a \\ & + \beta b\times b + \gamma\beta c\times b + \gamma\beta sa\times b - \beta^2 b\times b - \beta^2 uc\times b \\ & + \beta ub\times c + \gamma\beta uc\times c + \gamma\beta usa\times c - \beta^2 ub\times c - \beta^2 u^2 c\times c \\ & - \alpha b\times a - \gamma\alpha c\times a - \gamma\alpha sa\times a + \beta\alpha b\times a + \beta\alpha uc\times a \\ & - \alpha tb\times b - \gamma\alpha tc\times b - \gamma\alpha tsa\times b + \beta\alpha tb\times b + \beta\alpha tuc\times b \end{aligned}$$

$$\begin{aligned} e\times d = {} & + b\times a + \gamma c\times a - \beta b\times a - \beta uc\times a \\ & + \gamma\beta c\times b + \gamma\beta sa\times b - \beta^2 uc\times b \\ & + \beta ub\times c + \gamma\beta usa\times c - \beta^2 ub\times c \\ & - \alpha b\times a - \gamma\alpha c\times a + \beta\alpha b\times a + \beta\alpha uc\times a \\ & - \gamma\alpha tc\times b - \gamma\alpha tsa\times b + \beta\alpha tuc\times b \end{aligned}$$

$$e\times d = + b\times a - \gamma a\times c - \beta b\times a + \beta u a\times c$$

$$+ \gamma\beta c\times b - \gamma\beta s b\times a - \beta^2 u c\times b$$

$$- \beta u c\times b + \gamma\beta u s a\times c + \beta^2 u c\times b$$

$$- \alpha b\times a + \gamma\alpha a\times c + \beta\alpha b\times a - \beta\alpha u a\times c$$

$$- \gamma\alpha t c\times b + \gamma\alpha t s b\times a + \beta\alpha t u c\times b$$

Wegen $4iA_{\Delta ABC} = b\times a = c\times b = a\times c$ **folgt :**

$$e\times d = 4iA_{\Delta ABC}\cdot\left[\begin{array}{l} + 1 - \gamma - \beta + \beta u \\ + \gamma\beta - \gamma\beta s - \beta^2 u \\ - \beta u + \gamma\beta u s + \beta^2 u \\ - \alpha + \gamma\alpha + \beta\alpha - \beta\alpha u \\ - \gamma\alpha t + \gamma\alpha t s + \beta\alpha t u \end{array}\right]$$

$$e\times d = 4iA_{\Delta ABC}\cdot\left[\begin{array}{l} + 1 - \gamma - \beta \\ + \gamma\beta - \gamma\beta s \\ + \gamma\beta u s \\ - \alpha + \gamma\alpha + \beta\alpha - \beta\alpha u \\ - \gamma\alpha t + \gamma\alpha t s + \beta\alpha t u \end{array}\right]$$

$$e\times d = 4iA_{\Delta ABC}\cdot\left[\begin{array}{r} 1 - \alpha - \beta - \gamma + \alpha\beta + \beta\gamma + \gamma\alpha - \alpha\beta u - \beta\gamma s - \gamma\alpha t \\ + \alpha\beta t u + \beta\gamma u s + \gamma\alpha s t \end{array}\right]$$

$$\frac{e\times d}{4i} = A_{\Delta ABC}\cdot\left[\begin{array}{r} 1 - \alpha - \beta - \gamma + \alpha\beta + \beta\gamma + \gamma\alpha - \alpha\beta u - \beta\gamma s - \gamma\alpha t \\ + \alpha\beta t u + \beta\gamma u s + \gamma\alpha s t \end{array}\right]$$

$$A_{\Delta DEF} = A_{\Delta ABC}\cdot\left[\begin{array}{r} 1 - \alpha - \beta - \gamma + \alpha\beta + \beta\gamma + \gamma\alpha - \alpha\beta u - \beta\gamma s - \gamma\alpha t \\ + \alpha\beta t u + \beta\gamma u s + \gamma\alpha s t \end{array}\right]$$

$$\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \left[1 - \alpha - \beta - \gamma + \alpha\beta + \beta\gamma + \gamma\alpha - \alpha\beta u - \beta\gamma s - \gamma\alpha t + \alpha\beta t u + \beta\gamma u s + \gamma\alpha s t\right]$$

$$\boxed{\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \left[1 - \alpha - \beta - \gamma + \alpha\beta(1-u+tu) + \beta\gamma(1-s+us) + \gamma\alpha(1-t+st)\right]}$$

Resubstitution :

$$\alpha := \frac{s}{[1-t][1-s] + s} \qquad \beta := \frac{t}{[1-u][1-t] + t} \qquad \gamma := \frac{u}{[1-s][1-u] + u}$$

$$\alpha = \frac{s}{[1-t+st]} \qquad \beta = \frac{t}{[1-u+tu]} \qquad \gamma = \frac{u}{[1-s+us]}$$

$$\alpha\beta = \frac{st}{[1-t+st][1-u+tu]} \qquad \beta\gamma = \frac{tu}{[1-utu][1-s+us]} \qquad \gamma\alpha = \frac{us}{[1-s+us][1-t+st]}$$

$$\alpha\beta[1-u+tu] = \frac{st}{[1-t+st]} \qquad \beta\gamma[1-s+us] = \frac{tu}{[1-u+tu]} \qquad \gamma\alpha[1-t+st] = \frac{us}{[1-s+us]}$$

$$\boxed{\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \left[1 - \frac{s}{[1-t+st]} - \frac{t}{[1-u+tu]} - \frac{u}{[1-s+us]} + \frac{st}{[1-t+st]} + \frac{tu}{[1-u+tu]} + \frac{us}{[1-s+us]}\right]}$$

$$\boxed{\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \left[1 - \frac{s(1-t)}{[1-t+st]} - \frac{t(1-u)}{[1-u+tu]} - \frac{u(1-s)}{[1-s+us]}\right]}$$

Spezialfall : $s = t = u$

$$\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \left[1 - 3\,\frac{t}{[1-t+t^2]} + 3\,\frac{t^2}{[1-t+t^2]}\right]$$

$$\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \left[\frac{1-t+t^2}{[1-t+t^2]} - 3\,\frac{t}{[1-t+t^2]} + 3\,\frac{t^2}{[1-t+t^2]}\right]$$

$$\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \frac{1-t+t^2-3t+3t^2}{[1-t+t^2]}$$

$$\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = \frac{1-4t+4t^2}{[1-t+t^2]}$$

$$\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = 4\,\frac{\frac{1}{4}-t+t^2}{[1-t+t^2]}$$

$$\boxed{\frac{A_{\Delta DEF}}{A_{\Delta ABC}} = 4\,\frac{\left[t-\frac{1}{2}\right]^2}{[1-t+t^2]}}$$