In- und Ankreise des Dreiecks

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Quellen :

Fehringer, Arno : Geometrie in der komplexen Zahlenebene; Manuskript September 2019

Fehringer, Arno : Der Satz von Ceva und seine Umkehrung; Manuskript August 2019

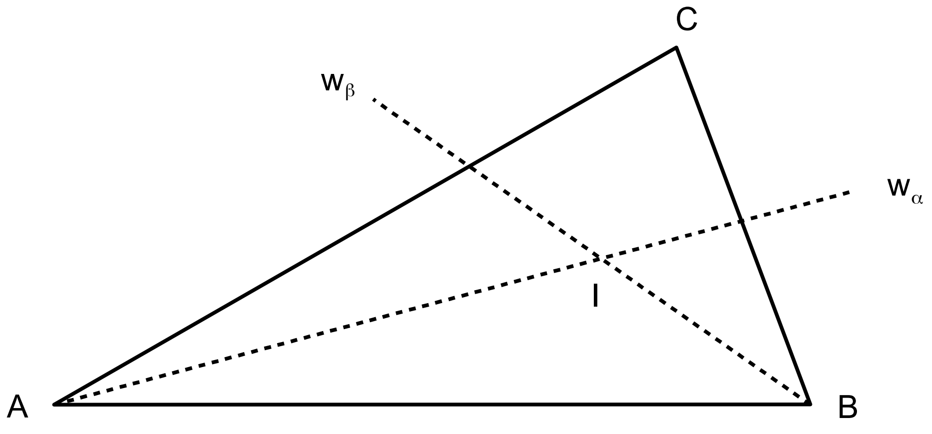
Zu finden auf meiner Homepage
Fehringer, Arno : https://mathematikgarten.hpage.com/

Polymath : The Nagel and Gergonne Points
https://polymathematics.typepad.com/polymath/the-.html

Willimzig, David : Dreiecksgeometrie
http://rho.math.uni-rostock.de/SemSkripte/Dreiecksgeometrie_Willimzig.pdf

Der Inkreis

Gegeben sei das Dreieck ΔABC .



Berechnung des Inkreis-Mittelpunktes I :

$$w_\alpha : \quad \left(\frac{\bar{c}}{|c|} - \frac{\bar{b}}{|b|}\right) z - \left(\frac{c}{|c|} - \frac{b}{|b|}\right) \bar{z} = \left(\frac{\bar{c}}{|c|} - \frac{\bar{b}}{|b|}\right) A - \left(\frac{c}{|c|} - \frac{b}{|b|}\right) \bar{A}$$

$$w_\beta : \quad \left(\frac{\bar{a}}{|a|} - \frac{\bar{c}}{|c|}\right) z - \left(\frac{a}{|a|} - \frac{c}{|c|}\right) \bar{z} = \left(\frac{\bar{a}}{|a|} - \frac{\bar{c}}{|c|}\right) B - \left(\frac{a}{|a|} - \frac{c}{|c|}\right) \bar{B}$$

$$w_\alpha : \quad \left(\frac{\bar{c}}{|c|} - \frac{\bar{b}}{|b|}\right) z - \left(\frac{c}{|c|} - \frac{b}{|b|}\right) \bar{z} = \left(\frac{\bar{c}}{|c|} - \frac{\bar{b}}{|b|}\right) A - \left(\frac{c}{|c|} - \frac{b}{|b|}\right) \bar{A}$$

$$w_\beta : \quad \left(\frac{\bar{a}}{|a|} - \frac{\bar{c}}{|c|}\right) z - \left(\frac{a}{|a|} - \frac{c}{|c|}\right) \bar{z} = \left(\frac{\bar{a}}{|a|} - \frac{\bar{c}}{|c|}\right)(A+c) - \left(\frac{a}{|a|} - \frac{c}{|c|}\right)(\bar{A}+\bar{c})$$

$$w_\alpha \cap w_\beta : \quad \boxed{I = \frac{Z}{N}}$$

$$N = \left(\frac{\bar{c}}{|c|} - \frac{\bar{b}}{|b|}\right)\left(\frac{a}{|a|} - \frac{c}{|c|}\right) - \left(\frac{c}{|c|} - \frac{b}{|b|}\right)\left(\frac{\bar{a}}{|a|} - \frac{\bar{c}}{|c|}\right)$$

$$N = \frac{a\bar{c}}{|a||c|} - \frac{\bar{c}c}{|c||c|} - \frac{\bar{b}a}{|b||a|} + \frac{c\bar{b}}{|c||b|} - \frac{\bar{a}c}{|a||c|} + \frac{\bar{c}c}{|c||c|} + \frac{b\bar{a}}{|b||a|} - \frac{c\bar{b}}{|c||b|}$$

$$N = \frac{a\bar{c}}{|a||c|} - \frac{\bar{b}a}{|b||a|} + \frac{c\bar{b}}{|c||b|} - \frac{\bar{a}c}{|a||c|} + \frac{b\bar{a}}{|b||a|} - \frac{\bar{c}b}{|c||b|}$$

$$N = -\frac{\bar{a}c}{|a||c|} + \frac{a\bar{c}}{|a||c|} - \frac{\bar{b}a}{|b||a|} + \frac{b\bar{a}}{|b||a|} + \frac{c\bar{b}}{|c||b|} - \frac{\bar{c}b}{|c||b|}$$

$$N = -\frac{\bar{a}c - a\bar{c}}{|a||c|} - \frac{\bar{b}a - b\bar{a}}{|b||a|} - \frac{\bar{c}b - c\bar{b}}{|c||b|}$$

Wegen $\bar{a}c - a\bar{c} = \bar{b}a - b\bar{a} = \bar{c}b - c\bar{b} = -4iA_{\Delta ABC}$ **folgt :**

$$N = 4iA_{\Delta ABC}\left[\frac{1}{|a||c|} + \frac{1}{|b||a|} + \frac{1}{|c||b|}\right]$$

$$N = 4iA_{\Delta ABC}\left[\frac{1}{|a||c|} + \frac{1}{|b||a|} + \frac{1}{|c||b|}\right]$$

$$N = \frac{4iA_{\Delta ABC}}{|a||b||c|}\left[|b| + |c| + |a|\right]$$

$$\boxed{N = \frac{4iA_{\Delta ABC}}{|a||b||c|}\left[|a| + |b| + |c|\right]}$$

$$Z = \left[\left(\frac{\bar{c}}{|c|}-\frac{\bar{b}}{|b|}\right)A - \left(\frac{c}{|c|}-\frac{b}{|b|}\right)\bar{A}\right]\left(\frac{a}{|a|}-\frac{c}{|c|}\right) - \left(\frac{c}{|c|}-\frac{b}{|b|}\right)\left[\left(\frac{\bar{a}}{|a|}-\frac{\bar{c}}{|c|}\right)(A+c) - \left(\frac{a}{|a|}-\frac{c}{|c|}\right)(\bar{A}+\bar{c})\right]$$

$$Z = \left[\left(\frac{\bar{c}}{|c|}-\frac{\bar{b}}{|b|}\right)A - \cancel{\left(\frac{c}{|c|}-\frac{b}{|b|}\right)\bar{A}}\right]\left(\frac{a}{|a|}-\frac{c}{|c|}\right) - \left(\frac{c}{|c|}-\frac{b}{|b|}\right)\left[\left(\frac{\bar{a}}{|a|}-\frac{\bar{c}}{|c|}\right)(A+c) - \left(\frac{a}{|a|}-\frac{c}{|c|}\right)(\cancel{\bar{A}}+\bar{c})\right]$$

$$Z = \left(\frac{\bar{c}}{|c|}-\frac{\bar{b}}{|b|}\right)\left(\frac{a}{|a|}-\frac{c}{|c|}\right)A - \left(\frac{c}{|c|}-\frac{b}{|b|}\right)\left(\frac{\bar{a}}{|a|}-\frac{\bar{c}}{|c|}\right)A - \left(\frac{c}{|c|}-\frac{b}{|b|}\right)\left(\frac{\bar{a}}{|a|}-\frac{\bar{c}}{|c|}\right)c + \left(\frac{c}{|c|}-\frac{b}{|b|}\right)\left(\frac{a}{|a|}-\frac{c}{|c|}\right)\bar{c}$$

$$Z = \left(\frac{a\bar{c}}{|a||c|} - \cancel{\frac{c\bar{c}}{|c||c|}} - \frac{\bar{b}a}{|b||a|} + \frac{c\bar{b}}{|c||b|}\right)A + \left(-\frac{\bar{a}c}{|a||c|} + \cancel{\frac{c\bar{c}}{|c||c|}} + \frac{b\bar{a}}{|b||a|} - \frac{\bar{c}b}{|c||b|}\right)A$$

$$+ \left(-\frac{\bar{a}c}{|a||c|} + \cancel{\frac{c\bar{c}}{|c||c|}} + \frac{b\bar{a}}{|b||a|} - \frac{\bar{c}b}{|c||b|}\right)c + \left(\frac{ac}{|a||c|} - \cancel{\frac{cc}{|c||c|}} - \frac{ba}{|b||a|} + \frac{cb}{|c||b|}\right)\bar{c}$$

$$Z = \left(\frac{a\bar{c}}{|a||c|} - \frac{\bar{b}a}{|b||a|} + \frac{c\bar{b}}{|c||b|}\right)A + \left(-\frac{\bar{a}c}{|a||c|} + \frac{b\bar{a}}{|b||a|} - \frac{\bar{c}b}{|c||b|}\right)A$$

$$+ \left(-\frac{\bar{a}c}{|a||c|} + \frac{b\bar{a}}{|b||a|} - \frac{\bar{c}b}{|c||b|}\right)c + \left(\frac{ac}{|a||c|} - \frac{ba}{|b||a|} + \frac{cb}{|c||b|}\right)\bar{c}$$

$$Z = \left[- \frac{\bar{a}c - a\bar{c}}{|a||c|} - \frac{\bar{c}b - c\bar{b}}{|c||b|} - \frac{\bar{b}a - b\bar{a}}{|b||a|} \right] A$$

$$- \frac{\bar{a}c^2}{|a||c|} + \frac{b\bar{a}c}{|b||a|} - \cancel{\frac{c\bar{c}b}{|c||b|}} + \frac{ac\bar{c}}{|a||c|} - \frac{ba\bar{c}}{|b||a|} + \cancel{\frac{c\bar{c}b}{|c||b|}}$$

$$Z = \left[- \frac{\bar{a}c - a\bar{c}}{|a||c|} - \frac{\bar{c}b - c\bar{b}}{|c||b|} - \frac{\bar{b}a - b\bar{a}}{|b||a|} \right] A$$

$$- \frac{\bar{a}c^2}{|a||c|} + \frac{b\bar{a}c}{|b||a|} + \frac{ac\bar{c}}{|a||c|} - \frac{ba\bar{c}}{|b||a|}$$

$$Z = \left[- \frac{\bar{a}c - a\bar{c}}{|a||c|} - \frac{\bar{c}b - c\bar{b}}{|c||b|} - \frac{\bar{b}a - b\bar{a}}{|b||a|} \right] A$$

$$- \frac{\bar{a}c^2}{|a||c|} + \frac{ac\bar{c}}{|a||c|} + \frac{b\bar{a}c}{|b||a|} - \frac{ba\bar{c}}{|b||a|}$$

$$Z = \left[- \frac{\bar{a}c-a\bar{c}}{|a||c|} - \frac{\bar{c}b-c\bar{b}}{|c||b|} - \frac{\bar{b}a-b\bar{a}}{|b||a|} \right] A - \frac{c}{|a||c|}\left(\bar{a}c-a\bar{c}\right) + \frac{b}{|b||a|}\left(\bar{a}c-a\bar{c}\right)$$

$$Z = \left[- \frac{\bar{a}c-a\bar{c}}{|a||c|} - \frac{\bar{c}b-c\bar{b}}{|c||b|} - \frac{\bar{b}a-b\bar{a}}{|b||a|} \right] A - \frac{c}{|a||c|}\left(\bar{a}c-a\bar{c}\right) + \frac{b}{|b||a|}\left(\bar{a}c-a\bar{c}\right)$$

Wegen $\bar{a}c-a\bar{c} = \bar{b}a-b\bar{a} = \bar{c}b-c\bar{b} = -4iA_{\Delta ABC}$ **folgt :**

$$Z = \left[\frac{4iA_{\Delta ABC}}{|a||c|} + \frac{4iA_{\Delta ABC}}{|c||b|} + \frac{4iA_{\Delta ABC}}{|b||a|} \right] A + \frac{c}{|a||c|} 4iA_{\Delta ABC} - \frac{b}{|b||a|} 4iA_{\Delta ABC}$$

$$Z = 4iA_{\Delta ABC} \left[\frac{1}{|a||c|} + \frac{1}{|c||b|} + \frac{1}{|b||a|} \right] A + \frac{4iA_{\Delta ABC}}{|a|} \left[\frac{c}{|c|} - \frac{b}{|b|} \right]$$

$$Z = \frac{4iA_{\Delta ABC}}{|a||b||c|} \left[|a| + |b| + |c| \right] A + \frac{4iA_{\Delta ABC}}{|a||b||c|} |b||c| \left[\frac{c}{|c|} - \frac{b}{|b|} \right]$$

$$Z = \frac{4iA_{\Delta ABC}}{|a||b||c|} \left[\left[|a| + |b| + |c| \right] A + |b||c| \left[\frac{c}{|c|} - \frac{b}{|b|} \right] \right]$$

$$Z = \frac{4iA_{\Delta ABC}}{|a||b||c|} \left[\, |a|A + |b|A + |c|A + |b|c - |c|b \, \right]$$

Wegen $c = B-A$, $b = A-C$ **folgt :**

$$Z = \frac{4iA_{\Delta ABC}}{|a||b||c|} \left[\, |a|A + |b|A + |c|A + |b|\left(B-A\right) - |c|\left(A-C\right) \, \right]$$

$$Z = \frac{4iA_{\Delta ABC}}{|a||b||c|}\left[\,|a|A + \cancel{|b|A} + \cancel{|c|A} + |b|B - \cancel{|b|A} - \cancel{|c|A} + |c|C\,\right]$$

$$\boxed{Z = \frac{4iA_{\Delta ABC}}{|a||b||c|}\left[\,|a|A + |b|B + |c|C\,\right]}$$

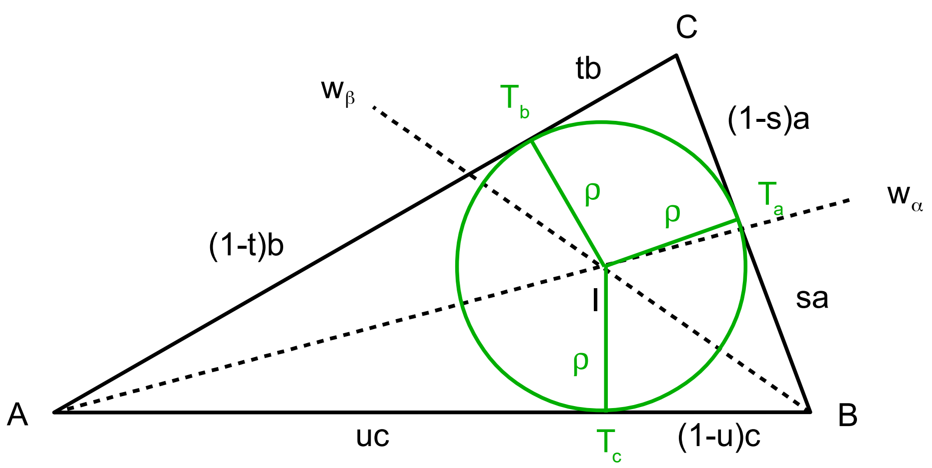
Damit folgt weiter :

$$I = \frac{Z}{N}$$

$$I = \frac{\dfrac{4iA_{\Delta ABC}}{|a||b||c|}\left[\,|a|A + |b|B + |c|C\,\right]}{\dfrac{4iA_{\Delta ABC}}{|a||b||c|}\left[|a| + |b| + |c|\right]}$$

$$\boxed{I = \frac{|a|A + |b|B + |c|C}{|a| + |b| + |c|}}$$ **Inkreis-Mittelpunkt**

Berechnung des Inkreisradius ρ **:**



$$A_{\Delta ABC} = \frac{1}{2}|a|\rho + \frac{1}{2}|b|\rho + \frac{1}{2}|c|\rho$$

$$A_{\Delta ABC} = \frac{1}{2}\left[|a|+|b|+|c|\right]\rho$$

$$\boxed{\rho = \frac{2A_{\Delta ABC}}{|a|+|b|+|c|}}$$

Wegen $\bar{a}c-a\bar{c} = \bar{b}a-b\bar{a} = \bar{c}b-c\bar{b} = -4iA_{\Delta ABC}$ **und** $a = C-B$, $b = A-C$, $c = B-A$ **folgt zunächst :**

$$-4iA_{\Delta ABC} = \bar{a}c-a\bar{c}$$

$$-4iA_{\Delta ABC} = (\bar{C}-\bar{B})(B-A) - (C-B)(\bar{B}-\bar{A})$$

$$-4iA_{\Delta ABC} = \bar{C}B - \bar{C}A - \bar{B}B + \bar{B}A - C\bar{B} + C\bar{A} + B\bar{B} - B\bar{A}$$

$$-4iA_{\Delta ABC} = \bar{C}B - \bar{C}A + \bar{B}A - C\bar{B} + C\bar{A} - B\bar{A}$$

$$-4iA_{\Delta ABC} = C\bar{A} - B\bar{A} + \bar{B}A - C\bar{B} + \bar{C}B - \bar{C}A$$

$$-4iA_{\Delta ABC} = \bar{A}(C-B) + \bar{B}(A-C) + \bar{C}(B-A)$$

$$\boxed{-4iA_{\Delta ABC} = \bar{A}a + \bar{B}b + \bar{C}c}$$

$$\boxed{A_{\Delta ABC} = -\frac{\bar{A}a + \bar{B}b + \bar{C}c}{4i}}$$

Damit folgt weiter :

$$\rho = \frac{2A_{\Delta ABC}}{|a|+|b|+|c|}$$

$$\rho = -\frac{2\frac{\bar{A}a + \bar{B}b + \bar{C}c}{4i}}{|a|+|b|+|c|}$$

$$\boxed{\rho = -\frac{\bar{A}a + \bar{B}b + \bar{C}c}{2i(|a|+|b|+|c|)}}$$ **Inkreisradius**

Abstände der Berührpunkte T_a , T_b , T_c **des Inkreises von den Ecken** B , C , A **:**

Der Lotfußpunkt des Lotes von I an die Seite a sei T_a . Man erhält ihn wie folgt :

BC : $\bar{a}z - a\bar{z} = \bar{a}B - a\bar{B}$
I : $\overline{-ia}z - -ia\bar{z} = \overline{-ia}I - -ia\bar{I}$

BC : $\bar{a}z - a\bar{z} = \bar{a}B - a\bar{B}$
I : $i\bar{a}z + ia\bar{z} = i\bar{a}I + ia\bar{I}$

BC : $\bar{a}z - a\bar{z} = \bar{a}B - a\bar{B}$
I : $\bar{a}z + a\bar{z} = \bar{a}I + a\bar{I}$

$$2\bar{a}z = \bar{a}(I+B) + a(\bar{I}-\bar{B})$$

$$z = \frac{\bar{a}(I+B) + a(\bar{I}-\bar{B})}{2\bar{a}}$$

$$\underline{T_a = \frac{\bar{a}(I+B) + a(\bar{I}-\bar{B})}{2\bar{a}}}$$

$$T_a-B = \frac{\bar{a}(I+B) + a(\bar{I}-\bar{B})}{2\bar{a}} - B$$

$$T_a-B = \frac{\bar{a}(I+B) + a(\bar{I}-\bar{B}) - 2\bar{a}B}{2\bar{a}}$$

$$T_a-B = \frac{\bar{a}I + \bar{a}B + a(\bar{I}-\bar{B}) - 2\bar{a}B}{2\bar{a}}$$

$$T_a-B = \frac{\bar{a}I - \bar{a}B + a(\bar{I}-\bar{B})}{2\bar{a}}$$

$$\underline{T_a-B = \frac{\bar{a}(I-B) + a(\bar{I}-\bar{B})}{2\bar{a}}} \quad , \quad \overline{T_a-B} = \frac{a(\bar{I}-\bar{B}) + \bar{a}(I-B)}{2a}$$

$$\underline{\overline{T_a-B} = \frac{\bar{a}(I-B) + a(\bar{I}-\bar{B})}{2a}}$$

$$|T_a-B|^2 = (T_a-b)\ \overline{T_a-B}$$

$$|T_a-B|^2 = \frac{\bar{a}(I-B) + a(\bar{I}-\bar{B})}{2\bar{a}} \cdot \frac{\bar{a}(I-B) + a(\bar{I}-\bar{B})}{2a}$$

$$\underline{|T_a-B|^2 = \frac{\left(\bar{a}(I-B) + a(\bar{I}-\bar{B})\right)^2}{4|a|^2}}$$

Nebenrechnung :

$$I = \frac{|a|A + |b|B + |c|C}{|a| + |b| + |c|}$$

$$I-B = \frac{|a|A + |b|B + |c|C}{|a| + |b| + |c|} - B$$

$$I-B = \frac{|a|A + |b|B + |c|C - |a|B - |b|B - |c|B}{|a| + |b| + |c|}$$

$$I-B = \frac{|a|A + |c|C - |a|B - |c|B}{|a| + |b| + |c|}$$

$$I-B = \frac{|c|C - |c|B - |a|B + |a|A}{|a| + |b| + |c|}$$

$$I-B = \frac{|c|(C-B) - |a|(B-A)}{|a| + |b| + |c|}$$

$$\underline{I-B = \frac{|c|a - |a|c}{|a| + |b| + |c|}} \quad , \quad \underline{\bar{I}-\bar{B} = \frac{|c|\bar{a} - |a|\bar{c}}{|a| + |b| + |c|}}$$

Jetzt folgt weiter :

$$\left|T_a-B\right|^2 = \frac{\left(\bar{a}\left(I-B\right) + a\left(\bar{I}-\bar{B}\right)\right)^2}{4\left|a\right|^2}$$

$$\left|T_a-B\right|^2 = \frac{\left(\bar{a}\ \left(\left|c\right|a - \left|a\right|c\right) + a\ \left(\left|c\right|\bar{a} - \left|a\right|\bar{c}\right)\right)^2}{4\left|a\right|^2\left(\left|a\right|+\left|b\right|+\left|c\right|\right)^2}$$

$$\left|T_a-B\right|^2 = \frac{\left(\left|a\right|^2\left|c\right| - \left|a\right|\bar{a}c + \left|a\right|^2\left|c\right| - \left|a\right|a\bar{c}\right)^2}{4\left|a\right|^2\left(\left|a\right|+\left|b\right|+\left|c\right|\right)^2}$$

$$\left|T_a-B\right|^2 = \frac{\left(2\left|a\right|^2\left|c\right| - \left|a\right|\left(\bar{a}c + a\bar{c}\right)\right)^2}{4\left|a\right|^2\left(\left|a\right|+\left|b\right|+\left|c\right|\right)^2}$$

$$\left|T_a-B\right|^2 = \frac{\left|a\right|^2\left(2\left|a\right|\left|c\right| - \left(\bar{a}c + a\bar{c}\right)\right)^2}{4\left|a\right|^2\left(\left|a\right|+\left|b\right|+\left|c\right|\right)^2}$$

$$\underline{\left|T_a-B\right|^2 = \frac{\left(2\left|a\right|\left|c\right| - \left(\bar{a}c + a\bar{c}\right)\right)^2}{4\left(\left|a\right|+\left|b\right|+\left|c\right|\right)^2}}$$

Nebenrechnung :

$$\bar{a}c + a\bar{c} = \left(-\bar{b}-\bar{c}\right)c + a\left(-\bar{a}-\bar{b}\right)$$

$$\bar{a}c + a\bar{c} = -\bar{b}c - \bar{c}c - a\bar{a} - a\bar{b}$$

$$\bar{a}c + a\bar{c} = -\bar{b}c - a\bar{b} - a\bar{a} - \bar{c}c$$

$$\bar{a}c + a\bar{c} = \bar{b}\left(-c-a\right) - a\bar{a} - \bar{c}c$$

$$\bar{a}c + a\bar{c} = \bar{b}b - a\bar{a} - \bar{c}c$$

$$\underline{\bar{a}c + a\bar{c} = \left|b\right|^2 - \left|a\right|^2 - \left|c\right|^2}$$

Damit folgt weiter :

$$\left|T_a-B\right|^2 = \frac{\left(2\left|a\right|\left|c\right| - \left(\bar{a}c + a\bar{c}\right)\right)^2}{4\left(\left|a\right|+\left|b\right|+\left|c\right|\right)^2}$$

$$\left|T_a-B\right|^2 = \frac{\left(2\left|a\right|\left|c\right| - \left|b\right|^2 + \left|a\right|^2 + \left|c\right|^2\right)^2}{4\left(\left|a\right|+\left|b\right|+\left|c\right|\right)^2}$$

$$\left|T_a-B\right|^2 = \frac{\left(\left|a\right|^2 + 2\left|a\right|\left|c\right| + \left|c\right|^2 - \left|b\right|^2\right)^2}{4\left(\left|a\right|+\left|b\right|+\left|c\right|\right)^2}$$

$$\left|T_a-B\right|^2 = \frac{\left(\left(\left|a\right|+\left|c\right|\right)^2 - \left|b\right|^2\right)^2}{4\left(\left|a\right|+\left|b\right|+\left|c\right|\right)^2}$$

$$\left|T_a-B\right|^2 = \frac{\left(\left(\left|a\right|+\left|c\right| - \left|b\right|\right)\left(\left|a\right|+\left|b\right| + \left|c\right|\right)\right)^2}{4\left(\left|a\right|+\left|b\right|+\left|c\right|\right)^2}$$

$$\left|T_a-B\right|^2 = \frac{\left(\,|a|+|c| \;-\; |b|\right)^2\;\left(|a|+|b| \;+\; |c|\right)^2}{4\left(|a|+|b|+|c|\right)^2}$$

$$\left|T_a-B\right|^2 = \frac{\left(\,|a|+|c| \;-\; |b|\right)^2}{4}$$

$$\left|T_a-B\right| = \frac{\left(\,|a|+|c| \;-\; |b|\right)}{2}$$

$$\left|T_a-B\right| = \frac{\left(\,|a|+|c|+|b| \;-\; 2|b|\right)}{2}$$

$$\boxed{\left|T_a-B\right| = \frac{|a|+|c|+|b|}{2} - |b|}\quad , \quad \textbf{wegen Symmetrie :}\quad \boxed{\left|T_a-C\right| = \frac{|a|+|c|+|b|}{2} - |c|}$$

Weiter aus Symmetriegründen:

$$\boxed{\left|T_b-C\right| = \frac{|a|+|c|+|b|}{2} - |c|}\qquad \boxed{\left|T_b-A\right| = \frac{|a|+|c|+|b|}{2} - |a|}$$

$$\boxed{\left|T_c-A\right| = \frac{|a|+|c|+|b|}{2} - |a|}\qquad \boxed{\left|T_c-B\right| = \frac{|a|+|c|+|b|}{2} - |b|}$$

Die Teilverhältnisse sind somit :

$$\frac{\left|T_a-B\right|}{\left|T_a-C\right|} = \frac{\dfrac{|a|+|c|+|b|}{2} - |b|}{\dfrac{|a|+|c|+|b|}{2} - |c|}$$

$$\boxed{\frac{\left|T_a-B\right|}{\left|T_a-C\right|} = \frac{S_{\Delta ABC} - |b|}{S_{\Delta ABC} - |c|}}\quad \text{mit}\quad S_{\Delta ABC} = \frac{|a|+|b|+|c|}{2}\quad \text{ist}$$

$$\boxed{\frac{\left|T_b-C\right|}{\left|T_b-A\right|} = \frac{S_{\Delta ABC} - |c|}{S_{\Delta ABC} - |a|}}$$

$$\boxed{\frac{\left|T_c-A\right|}{\left|T_c-B\right|} = \frac{S_{\Delta ABC} - |a|}{S_{\Delta ABC} - |b|}}$$

Das Produkt der Teilverhältnisse ist :

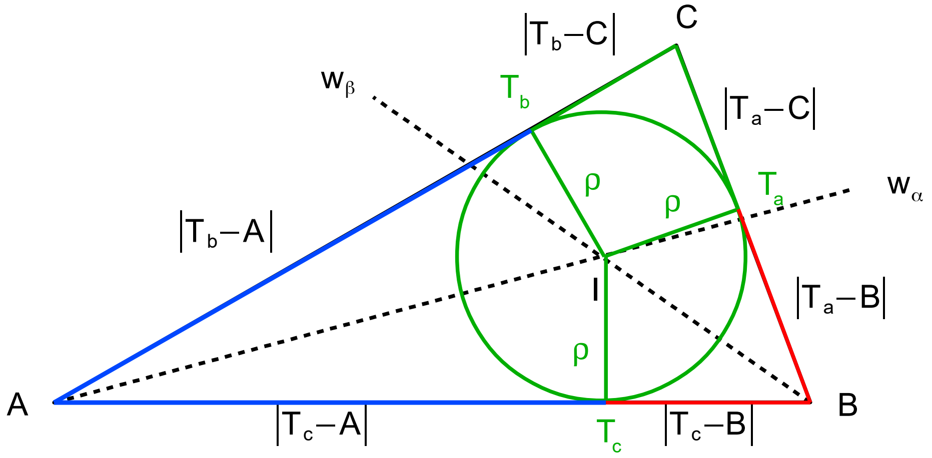
$$\frac{\left|T_a-B\right|}{\left|T_a-C\right|}\cdot\frac{\left|T_b-C\right|}{\left|T_b-A\right|}\cdot\frac{\left|T_c-A\right|}{\left|T_c-B\right|} = \frac{S_{\Delta ABC} - |b|}{S_{\Delta ABC} - |c|}\cdot\frac{S_{\Delta ABC} - |c|}{S_{\Delta ABC} - |a|}\cdot\frac{S_{\Delta ABC} - |a|}{S_{\Delta ABC} - |b|}$$

$$\boxed{\frac{\left|T_a-B\right|}{\left|T_a-C\right|}\cdot\frac{\left|T_b-C\right|}{\left|T_b-A\right|}\cdot\frac{\left|T_c-A\right|}{\left|T_c-B\right|} = 1}$$

Nach der **Umkehrung des Satzes von Ceva** schneiden sich die Geraden AT_a , BT_b , CT_c in einem Punkt , dem sogenannten **Gergonne -Punkt** !

Bemerkung :

Die Abstände der Berührpunkte T_a , T_b , T_c **des Inkreises von den Ecken** B , C , A **kann man schneller gewinnen unter Beachtung von Kongruenzen !**



$$\underline{|T_a-B|} + \underline{|T_a-C|} + \underline{|T_b-C|} + \underline{|T_b-A|} + \underline{|T_c-A|} + \underline{|T_c-B|} = |a| + |b| + |c|$$

$$\underline{2|T_a-B|} + \underline{2|T_b-C|} + \underline{2|T_c-A|} = |a| + |b| + |c|$$

$$|T_a-B| + |T_b-C| + |T_c-A| = \frac{|a| + |b| + |c|}{2}$$

$$|T_a-B| = \frac{|a| + |b| + |c|}{2} - \left(|T_b-C| + |T_c-A|\right)$$

$$|T_a-B| = \frac{|a| + |b| + |c|}{2} - \left(|T_b-C| + |T_c-A|\right)$$

Wegen $|T_b-C| + \underline{|T_c-A|} = |T_b-C| + \underline{|T_b-A|} = |b|$ **folgt :**

$$\boxed{|T_a-B| = \frac{|a| + |b| + |c|}{2} - |b|}$$

Zyklische Vertauschung liefert :

$$\boxed{|T_b-C| = \frac{|a| + |b| + |c|}{2} - |c|}$$

$$\boxed{|T_c-A| = \frac{|a| + |b| + |c|}{2} - |a|}$$

Berechnung des Gergonne-Punktes G

Im Skriptum **„Der Satz von Ceva und seine Umkehrung“** gibt es folgende Formel für diesen Schnittpunkt :

$$\boxed{G = \frac{xyz}{(1-y)z + xyz}\,A + \frac{xyz}{(1-z)x + xyz}\,B + \frac{xyz}{(1-x)y + xyz}\,C}$$

mit $x,y,z \in (0;1)$ und $T_a - B = xa$, $T_b - C = yb$, $T_c - A = zc$

Die Parameter x,y,z erhält man nach dem Vorigen wie folgt :

$$|T_a - B| = S_{\Delta ABC} - |b| = x|a|$$

$$|T_b - C| = S_{\Delta ABC} - |c| = y|b|$$

$$|T_c - A| = S_{\Delta ABC} - |a| = z|c|$$

$$\boxed{x = \frac{S_{\Delta ABC} - |b|}{|a|}}\ ,\quad \boxed{1-x = \frac{S_{\Delta ABC} - |c|}{|a|}}$$

$$\boxed{y = \frac{S_{\Delta ABC} - |c|}{|b|}}\ ,\quad \boxed{1-y = \frac{S_{\Delta ABC} - |a|}{|b|}}$$

$$\boxed{z = \frac{S_{\Delta ABC} - |a|}{|c|}}\ ,\quad \boxed{1-z = \frac{S_{\Delta ABC} - |b|}{|c|}}$$

Mit $S := S_{\Delta ABC} = \dfrac{|a|+|b|+|c|}{2}$ folgt $xyz = \dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}$ und

$$(1-y)z = \frac{(S-|a|)^2}{|b||c|}\ ,\quad (1-z)x = \frac{(S-|b|)^2}{|a||c|}\ ,\quad (1-x)y = \frac{(S-|c|)^2}{|a||b|}$$

$$\frac{xyz}{(1-y)z + xyz} = \frac{\dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}}{\dfrac{(S-|a|)^2}{|b||c|} + \dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}} = \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|a|)^2|a| + (S-|a|)(S-|b|)(S-|c|)}$$

$$\frac{xyz}{(1-z)x + xyz} = \frac{\dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}}{\dfrac{(S-|b|)^2}{|a||c|} + \dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}} = \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|b|)^2|b| + (S-|a|)(S-|b|)(S-|c|)}$$

$$\frac{xyz}{(1-x)y + xyz} = \frac{\dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}}{\dfrac{(S-|c|)^2}{|a||b|} + \dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}} = \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|c|)^2|c| + (S-|a|)(S-|b|)(S-|c|)}$$

$$G = \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|a|)^2|a| + (S-|a|)(S-|b|)(S-|c|)}\,A$$

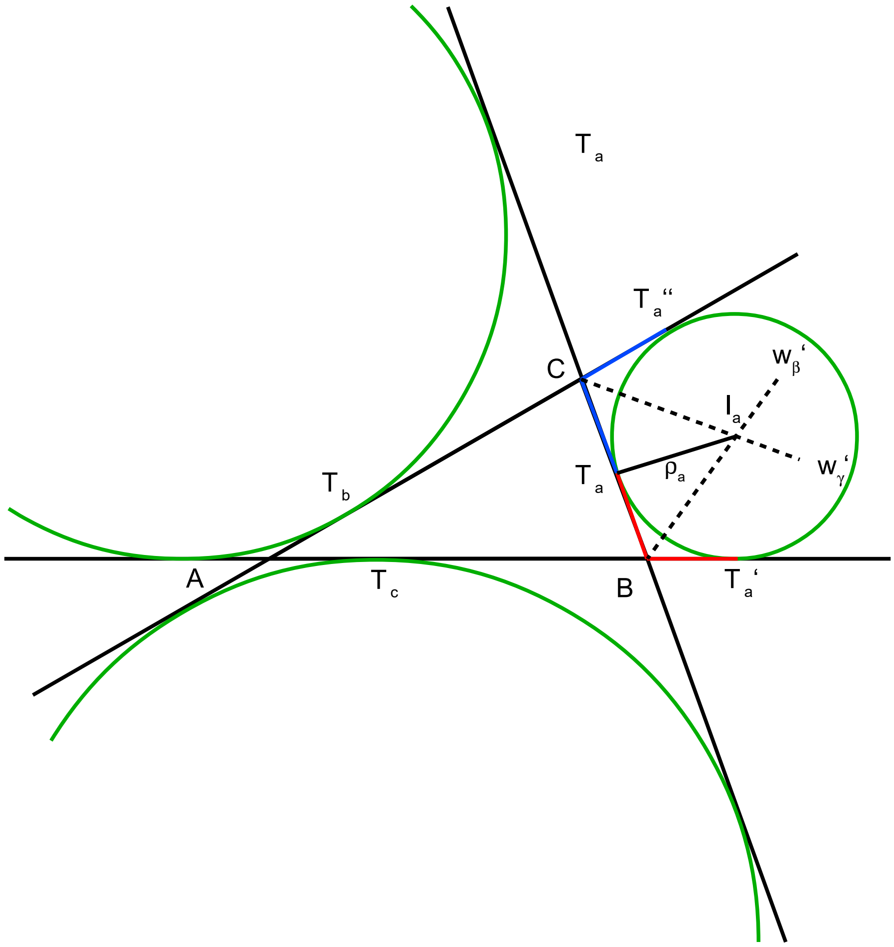
$$+ \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|b|)^2|b| + (S-|a|)(S-|b|)(S-|c|)}\,B$$

$$+ \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|c|)^2|c| + (S-|a|)(S-|b|)(S-|c|)}\,C$$

Gergonne-Punkt

Die Ankreise

Gegeben sei das Dreieck ΔABC und die Ankreise.



Wegen $\underline{|T_a'-B| = |T_a-B|}$, $\underline{|T_a''-C| = |T_a-C|}$ gilt :

$$\underline{|T_a'-A|} = \underline{|T_a'-B|} + |c| = \underline{|T_a-B|} + |c| \qquad \underline{|T_a''-A|} = \underline{|T_a''-C|} + |b| = \underline{|T_a-C|} + |b|$$

Wegen $\underline{|T_a'-A| = |T_a''-A|}$ folgt :

$$|T_a'-A| + |T_a''-A| = \underline{|T_a-B|} + |c| + \underline{|T_a-C|} + |b|$$

Wegen $\underline{|T_a-B| + |T_a-C|} = |a|$ folgt :

$$|T_a'-A| + |T_a''-A| = |a| + |b| + |c|$$

$$2|T_a'-A| = |a| + |b| + |c| \qquad 2|T_a''-A| = |a| + |b| + |c|$$

$$|T_a'-A| = \frac{|a| + |b| + |c|}{2} \qquad |T_a''-A| = \frac{|a| + |b| + |c|}{2}$$

$$|T_a-B| + |c| = \frac{|a| + |b| + |c|}{2} \qquad |T_a-C| + |b| = \frac{|a| + |b| + |c|}{2}$$

$$|T_a-B| = \frac{|a| + |b| + |c|}{2} - |c| \qquad |T_a-C| = \frac{|a| + |b| + |c|}{2} - |b|$$

$$\boxed{|T_a - B| = S_{\Delta ABC} - |c|} \qquad \boxed{|T_a - C| = S_{\Delta ABC} - |b|}$$

Zyklische Vertauschung liefert :

$$\boxed{|T_b - C| = S_{\Delta ABC} - |a|} \qquad \boxed{|T_b - A| = S_{\Delta ABC} - |c|}$$

$$\boxed{|T_c - A| = S_{\Delta ABC} - |b|} \qquad \boxed{|T_c - B| = S_{\Delta ABC} - |a|}$$

Die Teilverhältnisse sind somit :

$$\frac{|T_a - B|}{|T_a - C|} = \frac{S_{\Delta ABC} - |c|}{S_{\Delta ABC} - |b|} \qquad \frac{|T_b - C|}{|T_b - A|} = \frac{S_{\Delta ABC} - |a|}{S_{\Delta ABC} - |c|} \qquad \frac{|T_c - A|}{|T_c - B|} = \frac{S_{\Delta ABC} - |b|}{S_{\Delta ABC} - |a|}$$

Das Produkt der Teilverhältnisse ist :

$$\frac{|T_a - B|}{|T_a - C|} \cdot \frac{|T_b - C|}{|T_b - A|} \cdot \frac{|T_c - A|}{|T_c - B|} = \frac{S_{\Delta ABC} - |c|}{S_{\Delta ABC} - |b|} \cdot \frac{S_{\Delta ABC} - |a|}{S_{\Delta ABC} - |c|} \cdot \frac{S_{\Delta ABC} - |b|}{S_{\Delta ABC} - |a|}$$

$$\boxed{\frac{|T_a - B|}{|T_a - C|} \cdot \frac{|T_b - C|}{|T_b - A|} \cdot \frac{|T_c - A|}{|T_c - B|} = 1}$$

Nach der **Umkehrung des Satzes von Ceva** schneiden sich die Geraden AT_a , BT_b , CT_c in einem Punkt , dem sogenannten **Nagel-Punkt** !

Berechnung des Nagel-Punktes N

Im Skriptum **„Der Satz von Ceva und seine Umkehrung"** gibt es folgende Formel für diesen Schnittpunkt :

$$\boxed{N = \frac{xyz}{(1-y)z + xyz}A + \frac{xyz}{(1-z)x + xyz}B + \frac{xyz}{(1-x)y + xyz}C}$$

mit $x,y,z \in (0;1)$ und $T_a - B = xa$, $T_b - C = yb$, $T_c - A = zc$

Die Parameter x,y,z erhält man nach dem Vorigen wie folgt :

$$|T_a - B| = S_{\Delta ABC} - |c| = x|a|$$

$$|T_b - C| = S_{\Delta ABC} - |a| = y|b|$$

$$|T_c - A| = S_{\Delta ABC} - |b| = z|c|$$

$$\boxed{x = \frac{S_{\Delta ABC} - |c|}{|a|}} , \quad \boxed{1-x = \frac{S_{\Delta ABC} - |b|}{|a|}}$$

$$\boxed{y = \frac{S_{\Delta ABC} - |a|}{|b|}} , \quad \boxed{1-y = \frac{S_{\Delta ABC} - |c|}{|b|}}$$

$$\boxed{z = \frac{S_{\Delta ABC} - |b|}{|c|}} , \quad \boxed{1-z = \frac{S_{\Delta ABC} - |a|}{|c|}}$$

Mit $S := S_{\Delta ABC} = \dfrac{|a|+|b|+|c|}{2}$ folgt $xyz = \dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}$ und

$$(1-y)z = \frac{(S-|b|)(S-|c|)}{|b||c|} , \quad (1-z)x = \frac{(S-|c|)(S-|a|)}{|c||a|} , \quad (1-x)y = \frac{(S-|a|)(S-|b|)}{|a||b|}$$

$$\frac{xyz}{(1-y)z + xyz} = \frac{\dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}}{\dfrac{(S-|b|)(S-|c|)}{|b||c|} + \dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}} = \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|b|)(S-|c|)|a| + (S-|a|)(S-|b|)(S-|c|)}$$

$$\frac{xyz}{(1-z)x + xyz} = \frac{\dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}}{\dfrac{(S-|c|)(S-|a|)}{|c||a|} + \dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}} = \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|c|)(S-|a|)|b| + (S-|a|)(S-|b|)(S-|c|)}$$

$$\frac{xyz}{(1-x)y + xyz} = \frac{\dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}}{\dfrac{(S-|a|)(S-|b|)}{|a||b|} + \dfrac{(S-|a|)(S-|b|)(S-|c|)}{|a||b||c|}} = \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|a|)(S-|b|)|c| + (S-|a|)(S-|b|)(S-|c|)}$$

$$N = \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|b|)(S-|c|)|a| + (S-|a|)(S-|b|)(S-|c|)}\,A + \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|c|)(S-|a|)|b| + (S-|a|)(S-|b|)(S-|c|)}\,B + \frac{(S-|a|)(S-|b|)(S-|c|)}{(S-|a|)(S-|b|)|c| + (S-|a|)(S-|b|)(S-|c|)}\,C$$

Nagel-Punkt

Berechnung der Ankreis-Mittelpunkte I_a , I_b , I_c

$$w_\beta' : \quad \left(\frac{\bar{c}}{|c|} + \frac{\bar{a}}{|a|}\right) z - \left(\frac{c}{|c|} + \frac{a}{|a|}\right) \bar{z} = \left(\frac{\bar{c}}{|c|} + \frac{\bar{a}}{|a|}\right) B - \left(\frac{c}{|c|} + \frac{a}{|a|}\right) \bar{B}$$

$$w_\gamma' : \quad \left(\frac{\bar{a}}{|a|} + \frac{\bar{b}}{|b|}\right) z - \left(\frac{a}{|a|} + \frac{b}{|b|}\right) \bar{z} = \left(\frac{\bar{a}}{|a|} + \frac{\bar{b}}{|b|}\right) (B+a) - \left(\frac{a}{|a|} + \frac{b}{|b|}\right) (\bar{B}+\bar{a})$$

$$w_\beta' \cap w_\gamma' : \quad \boxed{I_a = \frac{Z}{N}}$$

$$N = \left(\frac{\bar{c}}{|c|} + \frac{\bar{a}}{|a|}\right)\left(\frac{a}{|a|} + \frac{b}{|b|}\right) - \left(\frac{c}{|c|} + \frac{a}{|a|}\right)\left(\frac{\bar{a}}{|a|} + \frac{\bar{b}}{|b|}\right)$$

$$N = \frac{a\bar{c}}{|a||c|} + \frac{\bar{c}b}{|c||b|} + \frac{\bar{a}a}{|a||a|} + \frac{b\bar{a}}{|b||a|} - \frac{\bar{a}c}{|a||c|} - \frac{c\bar{b}}{|c||b|} - \frac{a\bar{a}}{|a||a|} - \frac{\bar{b}a}{|b||a|}$$

$$N = \frac{a\bar{c}}{|a||c|} + \frac{\bar{c}b}{|c||b|} + \frac{b\bar{a}}{|b||a|} - \frac{\bar{a}c}{|a||c|} - \frac{c\bar{b}}{|c||b|} - \frac{\bar{b}a}{|b||a|}$$

$$N = -\frac{\bar{a}c}{|a||c|} + \frac{a\bar{c}}{|a||c|} + \frac{\bar{c}b}{|c||b|} - \frac{c\bar{b}}{|c||b|} - \frac{\bar{b}a}{|b||a|} + \frac{b\bar{a}}{|b||a|}$$

$$N = -\frac{\bar{a}c - a\bar{c}}{|a||c|} + \frac{\bar{c}b - c\bar{b}}{|c||b|} - \frac{\bar{b}a - b\bar{a}}{|b||a|}$$

Wegen $\bar{a}c - a\bar{c} = \bar{b}a - b\bar{a} = \bar{c}b - c\bar{b} = -4iA_{\Delta ABC}$ **folgt :**

$$N = 4iA_{\Delta ABC}\left[\frac{1}{|a||c|} - \frac{1}{|c||b|} + \frac{1}{|b||a|}\right]$$

$$\boxed{N = \frac{4iA_{\Delta ABC}}{|a||b||c|}\left[-|a| + |b| + |c|\right]}$$

$$Z = \left[\left(\frac{\bar{c}}{|c|}+\frac{\bar{a}}{|a|}\right)B - \cancel{\left(\frac{c}{|c|}+\frac{a}{|a|}\right)\bar{B}}\right]\left(\frac{a}{|a|}+\frac{b}{|b|}\right) - \left(\frac{c}{|c|}+\frac{a}{|a|}\right)\left[\left(\frac{\bar{a}}{|a|}+\frac{\bar{b}}{|b|}\right)(B+a) - \left(\frac{a}{|a|}+\frac{b}{|b|}\right)(\cancel{\bar{B}}+\bar{a})\right]$$

$$Z = \left[\left(\frac{\bar{c}}{|c|}+\frac{\bar{a}}{|a|}\right)B\right]\left(\frac{a}{|a|}+\frac{b}{|b|}\right) - \left(\frac{c}{|c|}+\frac{a}{|a|}\right)\left[\left(\frac{\bar{a}}{|a|}+\frac{\bar{b}}{|b|}\right)(B+a) - \left(\frac{a}{|a|}+\frac{b}{|b|}\right)\bar{a}\right]$$

$$Z = \left(\frac{\bar{c}}{|c|}+\frac{\bar{a}}{|a|}\right)\left(\frac{a}{|a|}+\frac{b}{|b|}\right)B - \left(\frac{c}{|c|}+\frac{a}{|a|}\right)\left(\frac{\bar{a}}{|a|}+\frac{\bar{b}}{|b|}\right)B - \left(\frac{c}{|c|}+\frac{a}{|a|}\right)\left(\frac{\bar{a}}{|a|}+\frac{\bar{b}}{|b|}\right)a + \left(\frac{c}{|c|}+\frac{a}{|a|}\right)\left(\frac{a}{|a|}+\frac{b}{|b|}\right)\bar{a}$$

$$Z = \left[\frac{a\bar{c}}{|a||c|} + \frac{\bar{c}b}{|c||b|} + \cancel{\frac{\bar{a}a}{|a||a|}} + \frac{b\bar{a}}{|b||a|}\right]B - \left[\frac{\bar{a}c}{|a||c|} + \frac{c\bar{b}}{|c||b|} + \cancel{\frac{a\bar{a}}{|a||a|}} + \frac{\bar{b}a}{|b||a|}\right]B$$

$$- \left[\cancel{\frac{\bar{a}c}{|a||c|}} + \frac{c\bar{b}}{|c||b|} + \cancel{\frac{a\bar{a}}{|a||a|}} + \frac{\bar{b}a}{|b||a|}\right]a + \left[\cancel{\frac{ac}{|a||c|}} + \frac{cb}{|c||b|} + \cancel{\frac{aa}{|a||a|}} + \frac{ba}{|b||a|}\right]\bar{a}$$

$$Z = \left[\frac{a\bar{c}}{|a||c|} + \frac{\bar{c}b}{|c||b|} + \frac{b\bar{a}}{|b||a|}\right]B - \left[\frac{\bar{a}c}{|a||c|} + \frac{c\bar{b}}{|c||b|} + \frac{\bar{b}a}{|b||a|}\right]B$$

$$- \left[\frac{c\bar{b}}{|c||b|} + \frac{\bar{b}a}{|b||a|}\right]a + \left[\frac{cb}{|c||b|} + \frac{ba}{|b||a|}\right]\bar{a}$$

$$Z = \left[\frac{a\bar{c}}{|a||c|} + \frac{\bar{c}b}{|c||b|} + \frac{b\bar{a}}{|b||a|}\right]B - \left[\frac{\bar{a}c}{|a||c|} + \frac{c\bar{b}}{|c||b|} + \frac{\bar{b}a}{|b||a|}\right]B$$

$$- \frac{c\bar{b}a}{|c||b|} - \frac{\bar{b}aa}{|b||a|}a + \frac{cb\bar{a}}{|c||b|} + \frac{ba\bar{a}}{|b||a|}$$

$$Z = \left[- \frac{\bar{a}c-a\bar{c}}{|a||c|} + \frac{\bar{c}b-c\bar{b}}{|c||b|} - \frac{\bar{b}a-b\bar{a}}{|b||a|}\right]B - c\,\frac{\bar{b}a-b\bar{a}}{|c||b|} - a\,\frac{\bar{b}a-b\bar{a}}{|b||a|}$$

Wegen $\bar{a}c-a\bar{c} = \bar{b}a-b\bar{a} = \bar{c}b-c\bar{b} = -4iA_{\Delta ABC}$ **folgt :**

$$Z = 4iA_{\Delta ABC}\left[\left(\frac{1}{|a||c|} - \frac{1}{|c||b|} + \frac{1}{|b||a|}\right)B + c\,\frac{1}{|c||b|} + a\,\frac{1}{|b||a|}\right]$$

$$Z = \frac{4iA_{\Delta ABC}}{|a||b||c|}\left[\left(|b| - |a| + |c|\right)B + |a|c + |c|a\right]$$

$$Z = \frac{4iA_{\Delta ABC}}{|a||b||c|}\left[\left(|b| - |a| + |c|\right)B + |a|\left(B-A\right) + |c|\left(C-B\right)\right]$$

$$Z = \frac{4iA_{\Delta ABC}}{|a||b||c|}\left[\ |b|B - \cancel{|a|B} + \cancel{|c|B} + \cancel{|a|B} - |a|A + |c|C - \cancel{|c|B}\ \right]$$

$$\boxed{Z = \frac{4iA_{\Delta ABC}}{|a||b||c|}\left[\ -|a|A + |b|B + |c|C\ \right]}$$

Damit folgt weiter :

$$I_a = \frac{Z}{N}$$

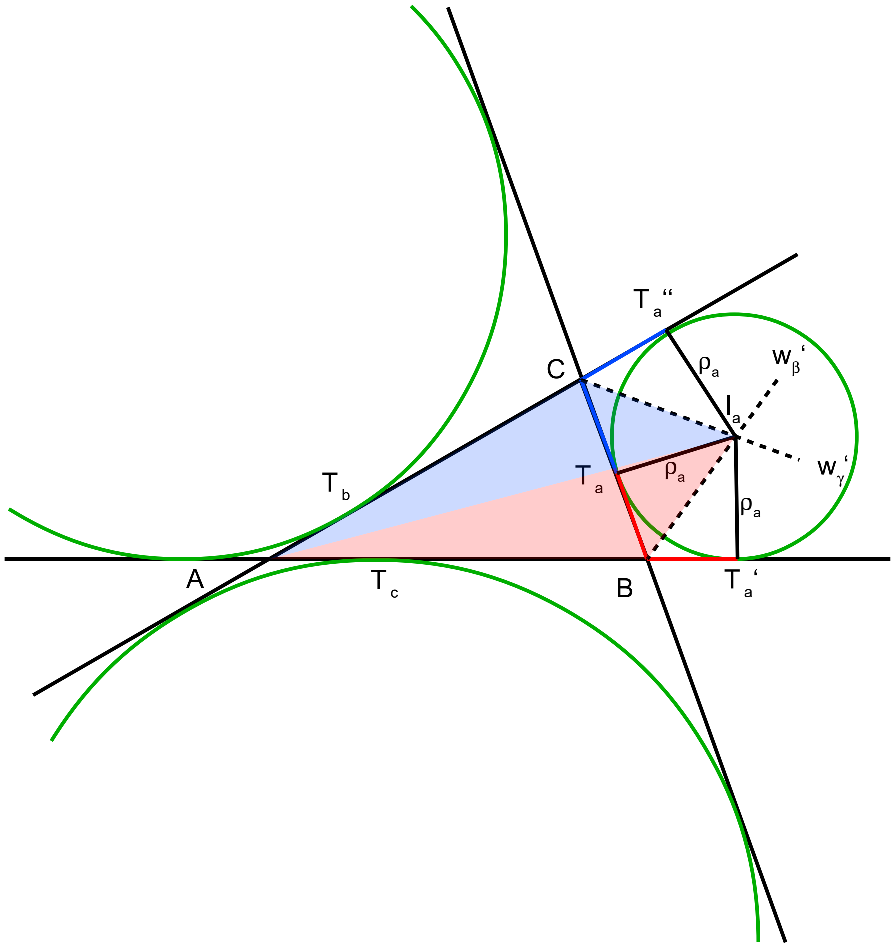
$$I_a = \frac{\dfrac{4iA_{\Delta ABC}}{|a||b||c|}\left[\ -|a|A + |b|B + |c|C\ \right]}{\dfrac{4iA_{\Delta ABC}}{|a||b||c|}\left[\ -|a| + |b| + |c|\ \right]}$$

$$\boxed{I_a = \frac{-|a|A + |b|B + |c|C}{-|a| + |b| + |c|}}$$

$$\boxed{I_b = \frac{|a|A - |b|B + |c|C}{|a| - |b| + |c|}}$$

$$\boxed{I_c = \frac{|a|A + |b|B - |c|C}{|a| + |b| - |c|}}$$

Berechnung der Ankreisradien ρ_a , ρ_b , ρ_c :



Flächenbetrachtung des Vierecks ABI_aC :

$$A_{\Delta ABC} + \frac{|a|\rho_a}{2} = \frac{|b|\rho_a}{2} + \frac{|c|\rho_a}{2}$$

$$S_{\Delta ABC}\,\rho + \frac{|a|\rho_a}{2} = \frac{|b|\rho_a}{2} + \frac{|c|\rho_a}{2}$$

$$S_{\Delta ABC}\,\rho = \frac{|b|\rho_a}{2} + \frac{|c|\rho_a}{2} - \frac{|a|\rho_a}{2}$$

$$S_{\Delta ABC}\,\rho = \frac{|a|\rho_a}{2} + \frac{|b|\rho_a}{2} + \frac{|c|\rho_a}{2} - 2\,\frac{|a|\rho_a}{2}$$

$$S_{\Delta ABC}\,\rho = S_{\Delta ABC}\,\rho_a - |a|\rho_a$$

$$S_{\Delta ABC}\,\rho = \left(S_{\Delta ABC} - |a|\right)\rho_a$$

$$\boxed{\rho_a = \frac{S_{\Delta ABC}}{\left(S_{\Delta ABC} - |a|\right)}\,\rho} \qquad \boxed{\rho_b = \frac{S_{\Delta ABC}}{\left(S_{\Delta ABC} - |b|\right)}\,\rho} \qquad \boxed{\rho_c = \frac{S_{\Delta ABC}}{\left(S_{\Delta ABC} - |c|\right)}\,\rho}$$

Wegen $\boxed{\rho = -\dfrac{\overline{A}a + \overline{B}b + \overline{C}c}{2i\left(|a|+|b|+|c|\right)}}$ **folgt :**

$$\boxed{\rho_a = -\frac{S_{\Delta ABC}}{\left(S_{\Delta ABC}-|a|\right)}\,\frac{\overline{A}a + \overline{B}b + \overline{C}c}{2i\left(|a|+|b|+|c|\right)}}$$

$$\boxed{\rho_b = -\frac{S_{\Delta ABC}}{\left(S_{\Delta ABC}-|b|\right)}\,\frac{\overline{A}a + \overline{B}b + \overline{C}c}{2i\left(|a|+|b|+|c|\right)}}$$

$$\boxed{\rho_c = -\frac{S_{\Delta ABC}}{\left(S_{\Delta ABC}-|c|\right)}\,\frac{\overline{A}a + \overline{B}b + \overline{C}c}{2i\left(|a|+|b|+|c|\right)}}$$

Zum Abschluss dieser Abhandlung noch zwei „schöne“ Zusammenhänge :

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} = \frac{\left(S_{\Delta ABC} - |a|\right)}{S_{\Delta ABC}\ \rho} + \frac{\left(S_{\Delta ABC} - |b|\right)}{S_{\Delta ABC}\ \rho} + \frac{\left(S_{\Delta ABC} - |c|\right)}{S_{\Delta ABC}\ \rho}$$

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} = \frac{\left(S_{\Delta ABC} - |a|\right) + \left(S_{\Delta ABC} - |b|\right) + \left(S_{\Delta ABC} - |c|\right)}{S_{\Delta ABC}\ \rho}$$

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} = \frac{3S_{\Delta ABC} - |a| - |b| - |c|}{S_{\Delta ABC}\ \rho}$$

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} = \frac{3S_{\Delta ABC} - 2S_{\Delta ABC}}{S_{\Delta ABC}\ \rho}$$

$$\boxed{\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} = \frac{1}{\rho}}$$

Wegen

$$I_a = \frac{-|a|A + |b|B + |c|C}{-|a| + |b| + |c|} = \frac{-|a|A + |b|B + |c|C}{2\left(S_{\Delta ABC} - |a|\right)}$$

$$I_b = \frac{|a|A - |b|B + |c|C}{|a| - |b| + |c|} = \frac{|a|A - |b|B + |c|C}{2\left(S_{\Delta ABC} - |b|\right)}$$

$$I_c = \frac{|a|A + |b|B - |c|C}{|a| + |b| - |c|} = \frac{|a|A + |b|B - |c|C}{2\left(S_{\Delta ABC} - |c|\right)}$$

folgt

$$\begin{aligned}\frac{1}{\rho_a} I_a + \frac{1}{\rho_b} I_b + \frac{1}{\rho_c} I_c = {} & \frac{\left(S_{\Delta ABC} - |a|\right)}{S_{\Delta ABC}\ \rho}\ \frac{-|a|A + |b|B + |c|C}{2\left(S_{\Delta ABC} - |a|\right)} \\ & + \frac{\left(S_{\Delta ABC} - |b|\right)}{S_{\Delta ABC}\ \rho}\ \frac{-|a|A + |b|B + |c|C}{2\left(S_{\Delta ABC} - |b|\right)} \\ & + \frac{\left(S_{\Delta ABC} - |c|\right)}{S_{\Delta ABC}\ \rho}\ \frac{|a|A + |b|B - |c|C}{2\left(S_{\Delta ABC} - |c|\right)}\end{aligned}$$

$$\begin{aligned}\frac{1}{\rho_a} I_a + \frac{1}{\rho_b} I_b + \frac{1}{\rho_c} I_c = {} & \frac{1}{S_{\Delta ABC}\ \rho}\ \frac{-|a|A + |b|B + |c|C}{2} \\ & + \frac{1}{S_{\Delta ABC}\ \rho}\ \frac{|a|A - |b|B + |c|C}{2} \\ & + \frac{1}{S_{\Delta ABC}\ \rho}\ \frac{|a|A + |b|B - |c|C}{2}\end{aligned}$$

$$\frac{1}{\rho_a} I_a + \frac{1}{\rho_b} I_b + \frac{1}{\rho_c} I_c = \frac{1}{2S_{\Delta ABC}\ \rho}\left(-|a|A + |b|B + |c|C\right) + \frac{1}{2S_{\Delta ABC}\ \rho}\left(|a|A - |b|B + |c|C\right) + \frac{1}{2S_{\Delta ABC}\ \rho}\left(|a|A + |b|B - |c|C\right)$$

$$\frac{1}{\rho_a} I_a + \frac{1}{\rho_b} I_b + \frac{1}{\rho_c} I_c = \frac{1}{2S_{\Delta ABC}\ \rho}\left(-|a|A + |b|B + |c|C\right) + \frac{1}{2S_{\Delta ABC}\ \rho}\left(|a|A - |b|B + |c|C\right) + \frac{1}{2S_{\Delta ABC}\ \rho}\left(|a|A + |b|B - |c|C\right)$$

$$\frac{1}{\rho_a} I_a + \frac{1}{\rho_b} I_b + \frac{1}{\rho_c} I_c = \frac{1}{2S_{\Delta ABC}\ \rho}\left(|a|A + |b|B + |c|C\right)$$

$$\frac{1}{\rho_a} I_a + \frac{1}{\rho_b} I_b + \frac{1}{\rho_c} I_c = \frac{1}{\rho}\,\frac{\left(|a|A + |b|B + |c|C\right)}{2S_{\Delta ABC}}$$

$$\frac{1}{\rho_a} I_a + \frac{1}{\rho_b} I_b + \frac{1}{\rho_c} I_c = \frac{1}{\rho}\,\frac{\left(|a|A + |b|B + |c|C\right)}{|a| + |b| + |c|}$$

$$\boxed{\frac{1}{\rho_a} I_a + \frac{1}{\rho_b} I_b + \frac{1}{\rho_c} I_c = \frac{1}{\rho} I}$$

Diese Formel wurde von **Hugo Wehrle**, einem Mathematik-Kollegen, aufgrund numerischer Berechnungen vermutet.