Gleichungen der Kegelschnitte

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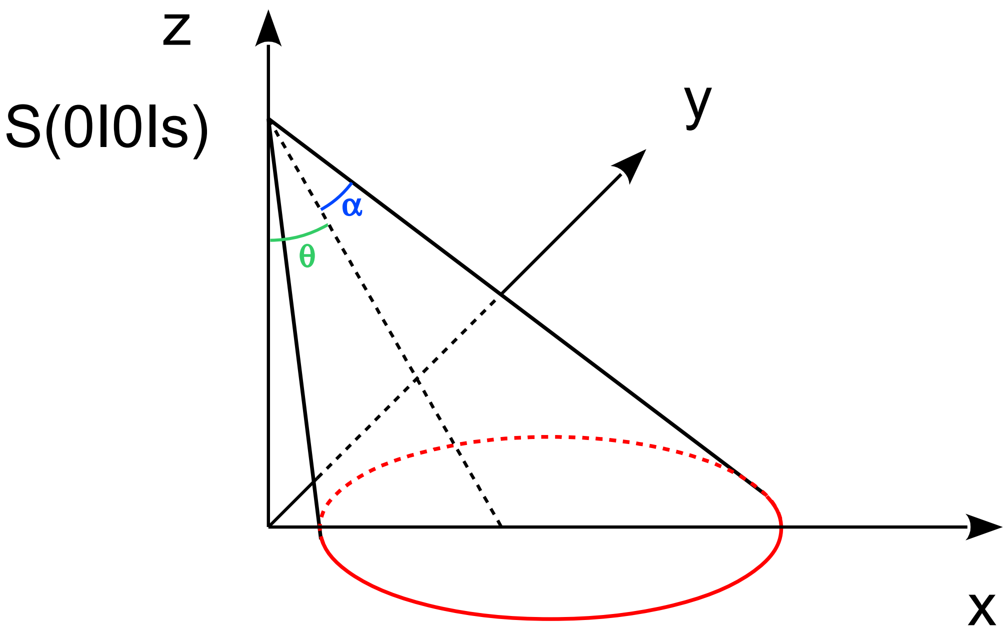
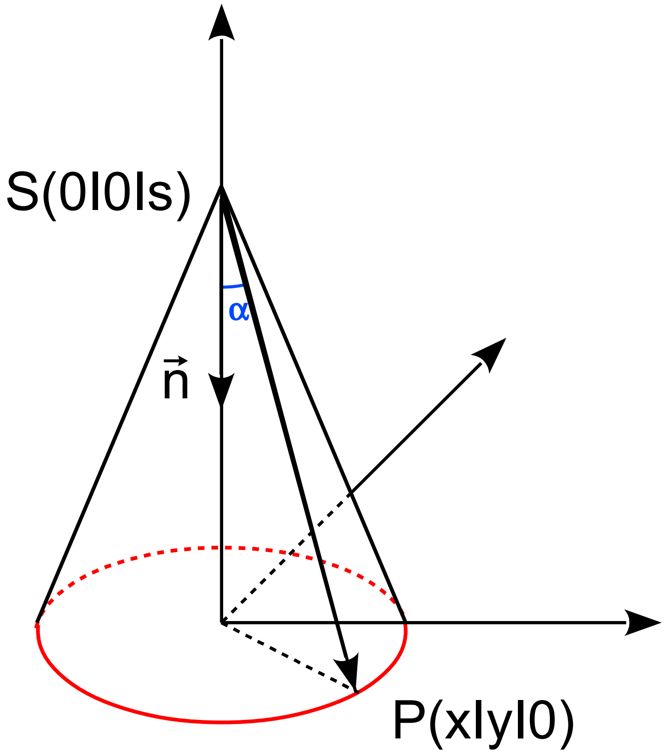
März 2022

Ein Kegel mit Spitze $S(0 \mid 0 \mid s)$ und halbem Öffnungswinkel α sei bezüglich der Symmetrieachse um den Winkel θ mit $0 \leq \theta < 90° + \alpha$ in Richtung x-Achse geneigt.

Gesucht ist die Gleichung der Schnittkurve des Kegels mit der xy-Ebene.



1. Fall : $\boxed{\theta = 0}$



$$\vec{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \qquad \overrightarrow{SP} = \begin{pmatrix} x \\ y \\ -s \end{pmatrix}$$

$$|\overrightarrow{SP}| = \sqrt{x^2+y^2+s^2}$$

$$|\overrightarrow{SP}| \cdot \cos(\alpha) = \vec{n} \cdot \overrightarrow{SP}$$

$$\sqrt{\left(x^2+y^2+s^2\right)}\cos(\alpha) = s$$

$$\left(x^2+y^2+s^2\right)\cos^2(\alpha) = s^2$$

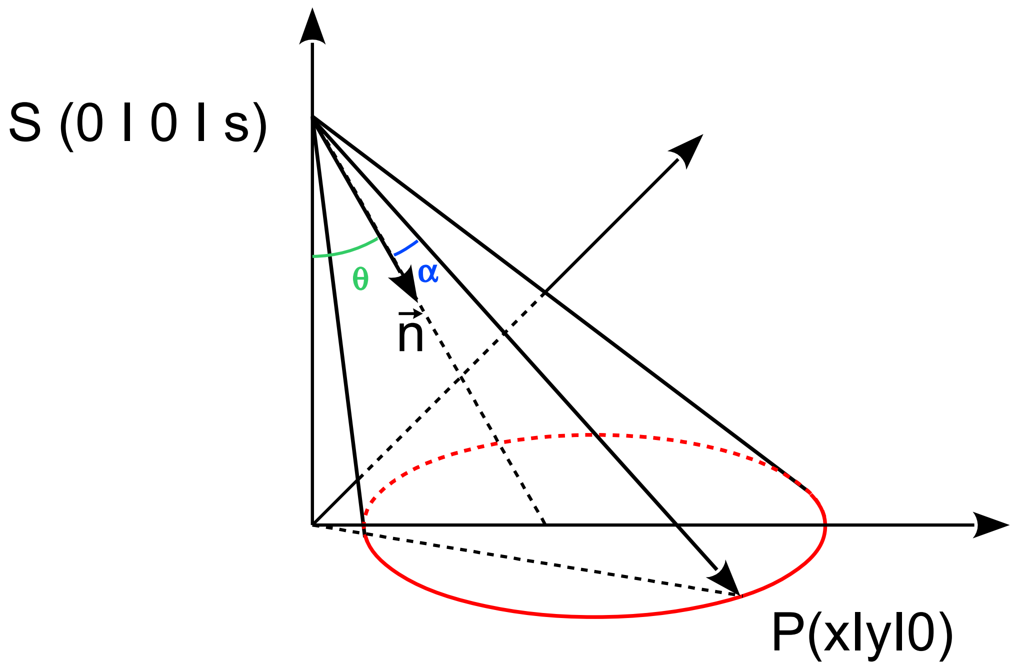
$$x^2+y^2+s^2 = \frac{s^2}{\cos^2(\alpha)}$$

$$x^2 + y^2 = \frac{s^2\left(1-\cos^2(\alpha)\right)}{\cos^2(\alpha)}$$

$$x^2 + y^2 = \left(s \cdot \tan(\alpha)\right)^2$$

Gleichung eines Kreises

2. Fall : $\theta+\alpha < \frac{\pi}{2} \Leftrightarrow \boxed{\theta < \frac{\pi}{2}-\alpha}$



$$\overrightarrow{SP} = \begin{pmatrix} x \\ y \\ -s \end{pmatrix} \qquad \vec{n} = \begin{pmatrix} \sin(\theta) \\ 0 \\ -\cos(\theta) \end{pmatrix}$$

$$|\overrightarrow{SP}| = \sqrt{x^2+y^2+s^2}$$

$$|\overrightarrow{SP}| \cdot \cos(\alpha) = \overrightarrow{SP} \cdot \vec{n}$$

$$\sqrt{x^2+y^2+s^2} \cdot \cos(\alpha) = \sin(\theta)\, x + s \cos(\theta)$$

$$\left(x^2+y^2+s^2\right) \cdot \cos^2(\alpha) = \sin^2(\theta)\, x^2 + 2s \sin(\theta) \cos(\theta)\, x + s^2 \cos^2(\theta)$$

$$\cos^2(\alpha) x^2 + \cos^2(\alpha) y^2 + s^2 \cos^2(\alpha) - \sin^2(\theta) x^2 - 2s \sin(\theta)\cos(\theta) x - s^2 \cos^2(\theta) = 0$$

$$\boxed{\left(\cos^2(\alpha)-\sin^2(\theta)\right) x^2 - 2s \sin(\theta)\cos(\theta) x + \cos^2(\alpha)\, y^2 + s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) = 0}$$

Gleichung der Schnittkurve

Bemerkung : Die Koeffizienten der Terme x^2 und y^2 sind positiv.

$$0 < \theta < \frac{\pi}{2}-\alpha \;\Rightarrow\; 0 < \sin(\theta) < \sin\left(\frac{\pi}{2}-\alpha\right) = \cos(\alpha)$$

$$\Rightarrow \sin^2(\theta) < \cos^2(\alpha)$$

$$\Rightarrow \boxed{0 < \cos^2(\alpha)-\sin^2(\theta)}$$

Berechnung der Schnittpunkte mit der x-Achse $y = 0$:

$$\left(\cos^2(\alpha)-\sin^2(\theta)\right)x^2 - 2s\,\sin(\theta)\cos(\theta)\,x + \cos^2(\alpha)\,y^2 + s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) = 0$$

$$y = 0$$

$$\left(\cos^2(\alpha)-\sin^2(\theta)\right)x^2 - 2s\,\sin(\theta)\cos(\theta)\,x + s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) = 0$$

Setze : $a := \cos^2(\alpha)-\sin^2(\theta) > 0$

$b := -2s\,\sin(\theta)\cos(\theta)$

$c := s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right)$

$$a x^2 + b x + c = 0$$

$$x_{1/2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad x_1 = \frac{-b - \sqrt{b^2-4ac}}{2a} \qquad x_2 = \frac{-b + \sqrt{b^2-4ac}}{2a}$$

Koordinatentransformation , Gleichung der Schnittkurve in den neuen Koordinaten:

$$x' = x - x_1 \quad \Leftrightarrow \quad x = x' + x_1 \qquad y' = y$$

$$\left(\cos^2(\alpha) - \sin^2(\theta)\right) x^2 - 2s\sin(\theta)\cos(\theta)\,x + \cos^2(\alpha)\,y^2 + s^2\left(\cos^2(\alpha) - \cos^2(\theta)\right) = 0$$

$$ax^2 + bx + \cos^2(\alpha)\,y^2 + c = 0$$

$$a\left(x' + x_1\right)^2 + b\left(x' + x_1\right) + \cos^2(\alpha)\,y^2 + c = 0$$

$$ax'^2 + 2ax_1x' + ax_1^2 + bx' + bx_1 + \cos^2(\alpha)\,y^2 + c = 0$$

$$ax'^2 + 2ax_1x' + bx' + \cos^2(\alpha)\,y^2 + \underbrace{ax_1^2 + bx_1 + c}_{0} = 0$$

$$ax'^2 + 2ax_1x' + bx' + \cos^2(\alpha)\,y^2 = 0$$

$$\underline{y^2 = \frac{-\left(2ax_1 + b\right)}{\cos^2(\alpha)}\,x' - \frac{a}{\cos^2(\alpha)}\,x'^2} \qquad \text{Schnittgleichung in } x', y$$

Wegen $x_1 = \dfrac{-b - \sqrt{b^2-4ac}}{2a}$ folgt

$$-\left(2ax_1+b\right) = \sqrt{b^2-4ac}$$

$$-\left(2ax_1+b\right) = \sqrt{4s^2\sin^2(\theta)\cos^2(\theta) - 4s^2\left(\cos^2(\alpha)-\sin^2(\theta)\right)\left(\cos^2(\alpha)-\cos^2(\theta)\right)}$$

$$\frac{-\left(2ax_1+b\right)}{\cos^2(\alpha)} = \frac{\sqrt{4s^2\sin^2(\theta)\cos^2(\theta) - 4s^2\left(\cos^2(\alpha)-\sin^2(\theta)\right)\left(\cos^2(\alpha)-\cos^2(\theta)\right)}}{\cos^2(\alpha)}$$

$$\frac{-\left(2ax_1+b\right)}{\cos^2(\alpha)} = \frac{\sqrt{4s^2\sin^2(\theta)\cos^2(\theta) - 4s^2\left(\cos^2(\alpha)-\sin^2(\theta)\right)\left(\cos^2(\alpha)-\cos^2(\theta)\right)}}{\cos^2(\alpha)}$$

$$\frac{-\left(2ax_1+b\right)}{\cos^2(\alpha)} = 2\ s\sqrt{\frac{\sin^2(\theta)}{\cos^2(\alpha)}\frac{\cos^2(\theta)}{\cos^2(\alpha)} - \left(1-\frac{\sin^2(\theta)}{\cos^2(\alpha)}\right)\left(1-\frac{\cos^2(\theta)}{\cos^2(\alpha)}\right)}$$

$$\frac{-\left(2ax_1+b\right)}{\cos^2(\alpha)} = 2\ s\sqrt{\frac{\cos^2(\theta)}{\cos^2(\alpha)} + \frac{\sin^2(\theta)}{\cos^2(\alpha)} - 1}$$

$$\frac{-(2ax_1+b)}{\cos^2(\alpha)} = 2\ s\sqrt{\frac{\cos^2(\theta)+\sin^2(\theta)-\cos^2(\alpha)}{\cos^2(\alpha)}}$$

$$\frac{-(2ax_1+b)}{\cos^2(\alpha)} = 2\ s\sqrt{\frac{\sin^2(\alpha)}{\cos^2(\alpha)}}$$

$$\frac{-(2ax_1+b)}{\cos^2(\alpha)} = 2\ s\ \tan(\alpha)$$

Es folgt weiter für die Schnittgleichung in x‘, y:

$$y^2 = \frac{-(2ax_1+b)}{\cos^2(\alpha)}\ x' - \frac{a}{\cos^2(\alpha)}x'^2$$

$$\boxed{y^2 = 2\ s\ \tan(\alpha)\ x' - \frac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}x'^2} \quad , \quad y^2 < 2\ s\ \tan(\alpha)\ x'$$

Für welche $x' = f$ **ist** $y = s\tan(\alpha)$ **?**

$$y^2 = 2\ s\tan(\alpha)\,x' - \frac{a}{\cos^2(\alpha)}x'^2 \quad , \qquad a := \cos^2(\alpha) - \sin^2(\theta) > 0$$

$$s^2\tan^2(\alpha) = 2\ s\tan(\alpha)\,f - \frac{a}{\cos^2(\alpha)}f^2$$

$$\frac{a}{\cos^2(\alpha)}f^2 - 2\ s\tan(\alpha)\,f + s^2\tan^2(\alpha) = 0$$

$$f_{1/2} = \frac{2s\tan(\alpha) - \sqrt{4s^2\tan^2(\alpha) - 4\dfrac{a\,s^2\tan^2(\alpha)}{\cos^2(\alpha)}}}{2\ \dfrac{a}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{2s\tan(\alpha) \pm \sqrt{4s^2\tan^2(\alpha)\left(1 - \dfrac{a}{\cos^2(\alpha)}\right)}}{2\ \dfrac{a}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{2\,s\tan(\alpha) \;\pm\; 2\,s\tan(\alpha)\sqrt{\dfrac{\cos^2(\alpha)-a}{\cos^2(\alpha)}}}{2\;\dfrac{a}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{2\,s\tan(\alpha) \;\pm\; 2\,s\tan(\alpha)\sqrt{\dfrac{\cos^2(\alpha)-\left(\cos^2(\alpha)-\sin^2(\theta)\right)}{\cos^2(\alpha)}}}{2\;\dfrac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{2\,s\tan(\alpha) \;\pm\; 2\,s\tan(\alpha)\sqrt{\dfrac{\sin^2(\theta)}{\cos^2(\alpha)}}}{2\;\dfrac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{2\,s\tan(\alpha) \;\pm\; 2\,s\tan(\alpha)\dfrac{\sin(\theta)}{\cos(\alpha)}}{2\;\dfrac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{s\tan(\alpha) \pm s\tan(\alpha)\dfrac{\sin(\theta)}{\cos(\alpha)}}{\dfrac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{s\tan(\alpha)\left(1 \pm \dfrac{\sin(\theta)}{\cos(\alpha)}\right)}{1-\dfrac{\sin^2(\theta)}{\cos^2(\alpha)}}$$

$$f_1 = \frac{s\tan(\alpha)}{1-\dfrac{\sin(\theta)}{\cos(\alpha)}} \qquad f_2 = \frac{s\tan(\alpha)}{1+\dfrac{\sin(\theta)}{\cos(\alpha)}}$$

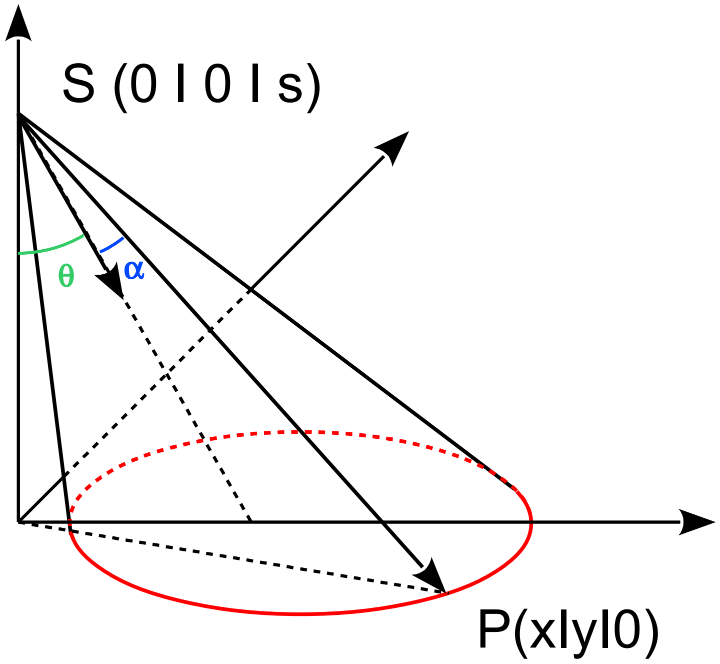
$$\underline{f_1 = \frac{s\sin(\alpha)}{\cos(\alpha)-\sin(\theta)}} \qquad \underline{f_2 = \frac{s\sin(\alpha)}{\cos(\alpha)+\sin(\theta)}}$$

Damit hat die Kurve die Punkte

$$\underline{F_1\left(\frac{s\sin(\alpha)}{\cos(\alpha)-\sin(\theta)} \;\mathrm{I}\; s\tan(\alpha)\right)} \qquad \underline{F_2\left(\frac{s\sin(\alpha)}{\cos(\alpha)+\sin(\theta)} \;\mathrm{I}\; s\tan(\alpha)\right)}$$

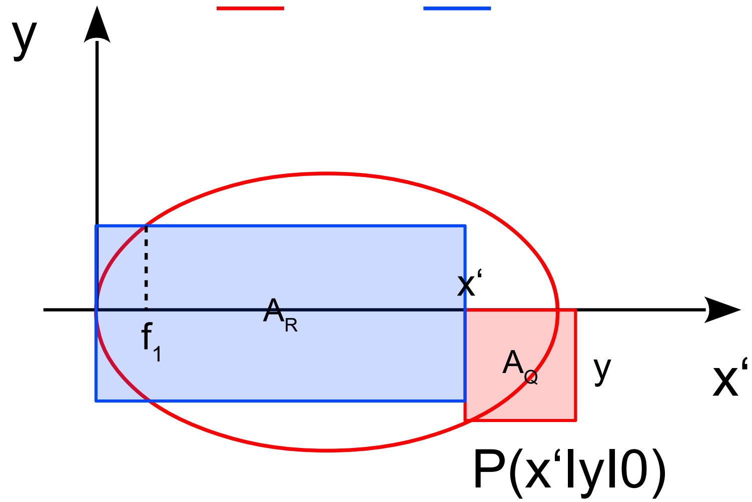
Geometrische Interpretation der Gleichung in x‘, y

$$\boxed{y^2 = 2\ s\tan(\alpha)\ x' - \frac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}x'^2}\quad,\qquad y^2 < 2\ s\tan(\alpha)\ x'$$



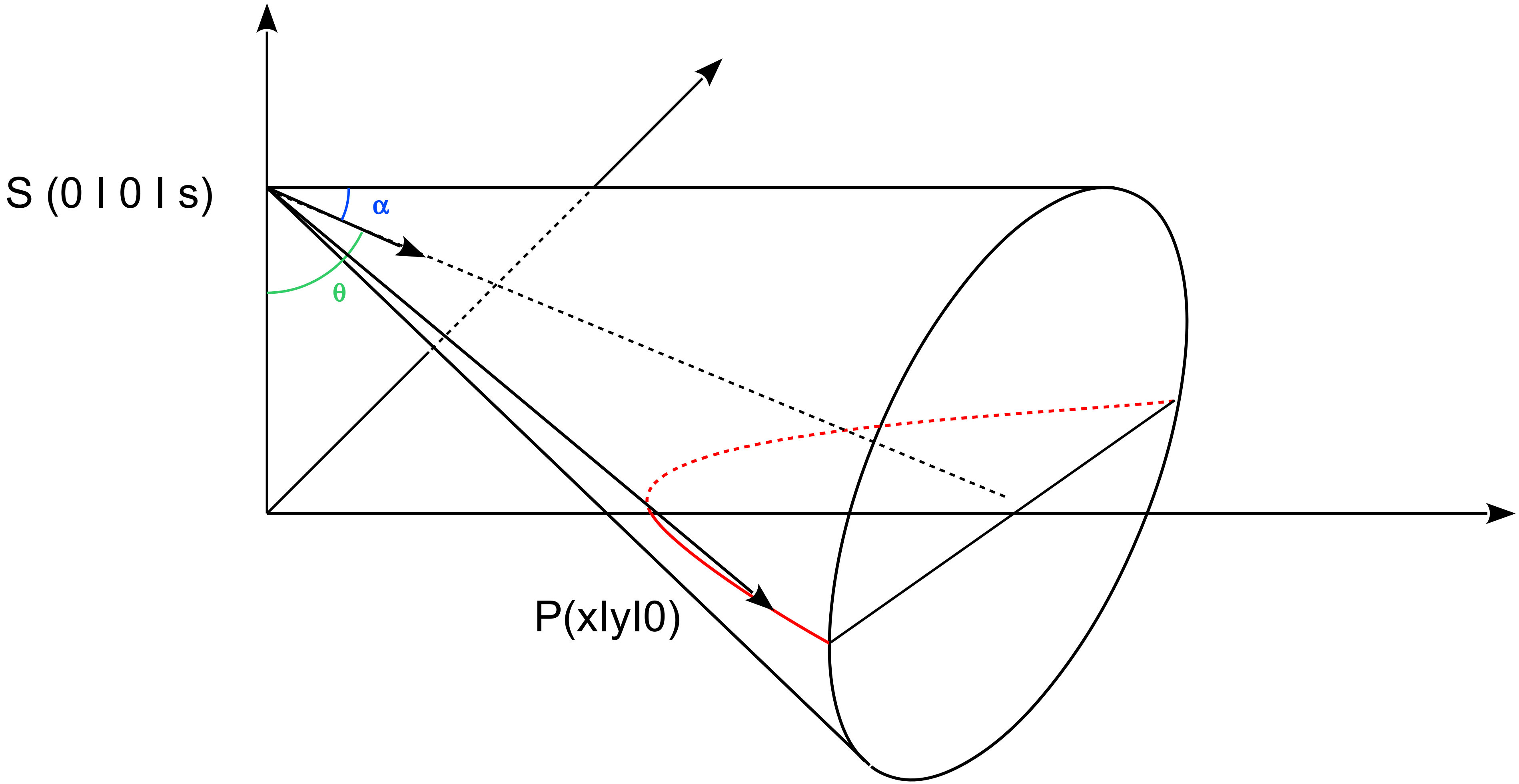
$y^2 < 2\,s\tan(\alpha)\,x'$

$A_Q < A_R$



Da der Flächeninhalt des Quadrats kleiner als der des Rechtecks ist, nennt man die Kurve **Ellipse**, von gr. ellipein = ελλιπειν = ermangeln .

3. Fall : $\theta+\alpha = \dfrac{\pi}{2} \quad \Leftrightarrow \quad \boxed{\theta = \dfrac{\pi}{2}-\alpha}$



Gleichung der Schnittkurve (siehe S. 4-5)

$$\left(\cos^2(\alpha)-\sin^2(\theta)\right)x^2 \;-\; 2s\,\sin(\theta)\cos(\theta)\,x \;+\; \cos^2(\alpha)\,y^2 \;+\; s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) \;=\; 0$$

Wegen $\theta = \frac{\pi}{2}-\alpha$ ist $\sin(\theta) = \sin\left(\frac{\pi}{2}-\alpha\right) = \cos(\theta)$, so dass der Koeffizient von x^2 verschwindet :

$$-2s\,\sin(\theta)\cos(\theta)\,x \;+\; \cos^2(\alpha)\,y^2 \;+\; s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) \;=\; 0$$

Berechnung der Schnittpunkte mit der x-Achse $y = 0$ **:**

$$-2s\,\sin(\theta)\cos(\theta)\,x \;+\; \cos^2(\alpha)\,y^2 \;+\; s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) \;=\; 0$$

$$y \;=\; 0$$

$$-2s\,\sin(\theta)\cos(\theta)x \;+\; s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) \;=\; 0$$

$$-2s\,\sin(\theta)\cos(\theta)x \;=\; -s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right)$$

$$x \;=\; \frac{-s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right)}{-2s\,\sin(\theta)\cos(\theta)}$$

$$\underline{x_0 \;=\; \frac{s\left(\cos^2(\alpha)-\cos^2(\theta)\right)}{2\sin(\theta)\cos(\theta)}}$$

Koordinatentransformation, Gleichung der Schnittkurve in den neuen Koordinaten :

$$x' \;=\; x-x_0 \;\;\Leftrightarrow\;\; x \;=\; x'+x_0 \qquad\qquad y' \;=\; y$$

$$-2s\,\sin(\theta)\cos(\theta)x \;+\; \cos^2(\alpha)y^2 \;+\; s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) \;=\; 0$$

$$-2s\,\sin(\theta)\cos(\theta)\left(x'+x_0\right) \;+\; \cos^2(\alpha)y^2 \;+\; s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) \;=\; 0$$

$$-2s\sin(\theta)\cos(\theta)x' - 2s\sin(\theta)\cos(\theta)x_0 + \cos^2(\alpha)y^2 + s^2\left(\cos^2(\alpha) - \cos^2(\theta)\right) = 0$$

$$-2s\sin(\theta)\cos(\theta)x' + \cos^2(\alpha)y^2 \underbrace{-2s\sin(\theta)\cos(\theta)x_0 + s^2\left(\cos^2(\alpha) - \cos^2(\theta)\right)}_{0} = 0$$

$$-2s\sin(\theta)\cos(\theta)x' + \cos^2(\alpha)y^2 = 0$$

$$\cos^2(\alpha)y^2 = 2s\sin(\theta)\cos(\theta)x'$$

$$\boxed{y^2 = 2\,\frac{s\sin(\theta)\cos(\theta)}{\cos^2(\alpha)}x'}$$

Für welche $x' = f$ **ist** $y = \dfrac{s\sin(\theta)\cos(\theta)}{\cos^2(\alpha)}$ **?**

$$y^2 = 2\,\frac{s\sin(\theta)\cos(\theta)}{\cos^2(\alpha)}x'$$

$$y^2 = 2\frac{s\sin(\theta)\cos(\theta)}{\cos^2(\alpha)}x'$$

$$y = \frac{s\sin(\theta)\cos(\theta)}{\cos^2(\alpha)}$$

$$\frac{s^2\sin^2(\theta)\cos^2(\theta)}{\cos^4(\alpha)} = 2\frac{s\sin(\theta)\cos(\theta)}{\cos^2(\alpha)}f$$

$$\frac{s\sin(\theta)\cos(\theta)}{\cos^2(\alpha)} = 2f$$

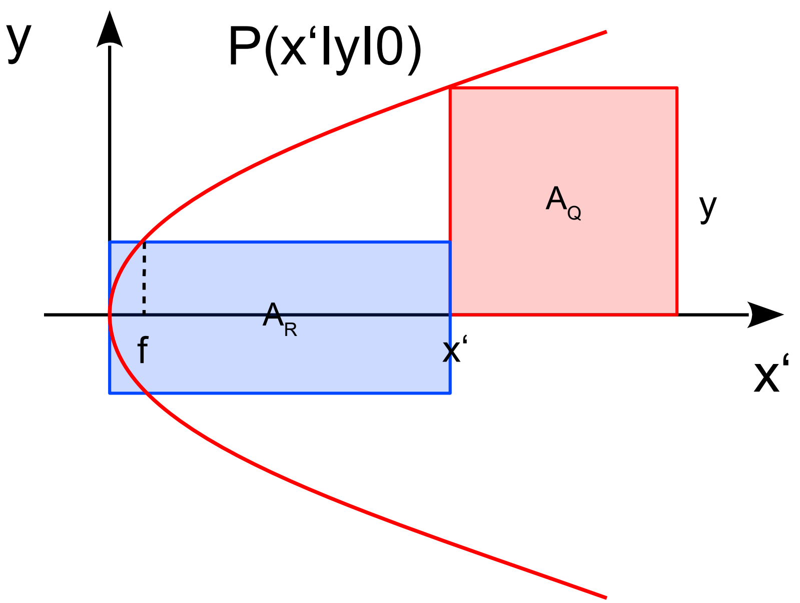
$$\underline{f = \frac{s\sin(\theta)\cos(\theta)}{2\cos^2(\alpha)}}$$

Damit hat die Kurve den Punkt

$$\underline{F\left(\frac{s\sin(\theta)\cos(\theta)}{2\cos^2(\alpha)} \;\mathrm{I}\; \frac{s\sin(\theta)\cos(\theta)}{\cos^2(\alpha)}\right)}$$

Geometrische Interpretation der Gleichung

$$\boxed{y^2 = 2\,\frac{s\,\sin(\theta)\cos(\theta)}{\cos^2(\alpha)}\,x'}$$

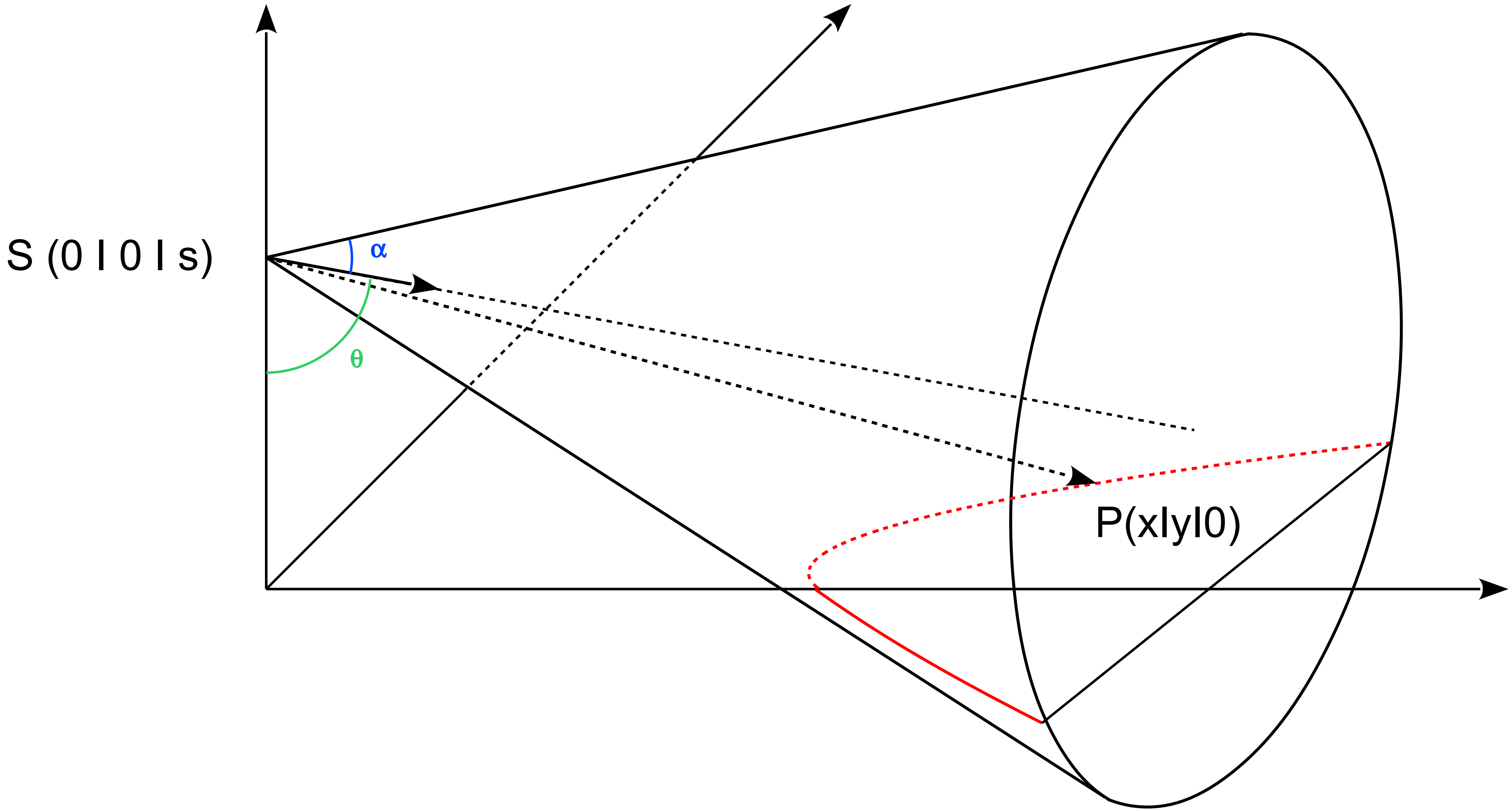


$$y^2 = 2\,\frac{s\,\sin(\theta)\cos(\theta)}{\cos^2(\alpha)}\,x'$$

$$\underline{A_Q} = \underline{A_R}$$

Da der Flächeninhalt des Quadrats gleich dem des Rechtecks ist, nennt man die Kurve **Parabel**, von gr. paraballein = παραβαλλειν = gleichkommen .

4. Fall : $\boxed{\dfrac{\pi}{2}-\alpha \;<\; \theta \;<\; \dfrac{\pi}{2}+\alpha}$



Gleichung der Schnittkurve (siehe S. 4-5)

$$\left(\cos^2(\alpha)-\sin^2(\theta)\right)x^2 \;-\; 2s\,\sin(\theta)\cos(\theta)\,x \;+\; \cos^2(\alpha)\,y^2 \;+\; s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) \;=\; 0$$

Wegen

$$0 < \frac{\pi}{2}-\alpha < \theta < \frac{\pi}{2}+\alpha$$

$$0 < \sin\left(\frac{\pi}{2}-\alpha\right) < \sin(\theta) < \sin\left(\frac{\pi}{2}+\alpha\right)$$

$$0 < \cos(\alpha) < \sin(\theta)$$

$$\cos^2(\alpha) < \sin^2(\theta)$$

$$\boxed{\cos^2(\alpha)-\sin^2(\theta) < 0}$$

haben die Koeffizienten von x^2 und y^2 unterschiedliche Vorzeichen .

Berechnung der Schnittpunkte mit der x-Achse $y = 0$:

$$\left(\cos^2(\alpha)-\sin^2(\theta)\right)x^2 - 2s\sin(\theta)\cos(\theta)x + \cos^2(\alpha)y^2 + s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) = 0$$

$$y = 0$$

$$\left(\cos^2(\alpha)-\sin^2(\theta)\right)x^2 - 2s\sin(\theta)\cos(\theta)x + s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) = 0$$

Setze : $a := \cos^2(\alpha)-\sin^2(\theta) < 0$

$b := -2s\sin(\theta)\cos(\theta)$

$c := s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right)$

$$ax^2 + bx + c = 0$$

$$x_{1/2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad x_1 = \frac{-b - \sqrt{b^2-4ac}}{2a} \qquad x_2 = \frac{-b + \sqrt{b^2-4ac}}{2a}$$

Koordinatentransformation, Gleichung der Schnittkurve in den neuen Koordinaten :

$$x' = x - x_2 \quad \Leftrightarrow \quad x = x' + x_2 \qquad y' = y$$

$$\left(\cos^2(\alpha) - \sin^2(\theta)\right)x^2 - 2s\sin(\theta)\cos(\theta)x + \cos^2(\alpha)y^2 + s^2\left(\cos^2(\alpha) - \cos^2(\theta)\right) = 0$$

$$ax^2 + bx + \cos^2(\alpha)y^2 + c = 0$$

$$a\left(x' + x_2\right)^2 + b\left(x' + x_2\right) + \cos^2(\alpha)y^2 + c = 0$$

$$ax'^2 + 2ax_2x' + ax_2^2 + bx' + bx_2 + \cos^2(\alpha)y^2 + c = 0$$

$$ax'^2 + 2ax_2x' + bx' + \cos^2(\alpha)y^2 + \underbrace{ax_2^2 + bx_2 + c}_{0} = 0$$

$$ax'^2 + 2ax_1x' + bx' + \cos^2(\alpha)y^2 = 0$$

$$\underline{y^2 = \frac{-\left(2ax_1 + b\right)}{\cos^2(\alpha)}x' - \frac{a}{\cos^2(\alpha)}x'^2}$$

Wegen $\quad x_1 = \dfrac{-b - \sqrt{b^2-4ac}}{2a}$ folgt

$$-\left(2ax_2+b\right) = \sqrt{b^2-4ac}$$

$$-\left(2ax_2+b\right) = \sqrt{4s^2\sin^2(\theta)\cos^2(\theta) - 4s^2\left(\cos^2(\alpha)-\sin^2(\theta)\right)\left(\cos^2(\alpha)-\cos^2(\theta)\right)}$$

$$\frac{-\left(2ax_2+b\right)}{\cos^2(\alpha)} = \frac{\sqrt{4s^2\sin^2(\theta)\cos^2(\theta) - 4s^2\left(\cos^2(\alpha)-\sin^2(\theta)\right)\left(\cos^2(\alpha)-\cos^2(\theta)\right)}}{\cos^2(\alpha)}$$

$$\frac{-\left(2ax_2+b\right)}{\cos^2(\alpha)} = \frac{\sqrt{4s^2\sin^2(\theta)\cos^2(\theta) - 4s^2\left(\cos^2(\alpha)-\sin^2(\theta)\right)\left(\cos^2(\alpha)-\cos^2(\theta)\right)}}{\cos^2(\alpha)}$$

$$\frac{-\left(2ax_2+b\right)}{\cos^2(\alpha)} = 2\ s\sqrt{\frac{\sin^2(\theta)}{\cos^2(\alpha)}\frac{\cos^2(\theta)}{\cos^2(\alpha)} - \left(1-\frac{\sin^2(\theta)}{\cos^2(\alpha)}\right)\left(1-\frac{\cos^2(\theta)}{\cos^2(\alpha)}\right)}$$

$$\frac{-\left(2ax_2+b\right)}{\cos^2(\alpha)} = 2\ s\sqrt{\frac{\cos^2(\theta)}{\cos^2(\alpha)} + \frac{\sin^2(\theta)}{\cos^2(\alpha)} - 1}$$

$$\frac{-\left(2ax_2+b\right)}{\cos^2(\alpha)} = 2\ s\sqrt{\frac{\cos^2(\theta)+\sin^2(\theta)-\cos^2(\alpha)}{\cos^2(\alpha)}}$$

$$\frac{-\left(2ax_2+b\right)}{\cos^2(\alpha)} = 2\ s\sqrt{\frac{\sin^2(\alpha)}{\cos^2(\alpha)}}$$

$$\frac{-\left(2ax_2+b\right)}{\cos^2(\alpha)} = 2\ s\ \tan(\alpha)$$

Es folgt weiter für die Schnittgleichung in x‘, y:

$$y^2 = \frac{-\left(2ax_2+b\right)}{\cos^2(\alpha)}\ x' - \frac{a}{\cos^2(\alpha)}x'^2$$

$$\boxed{y^2 = 2\ s\ \tan(\alpha)\ x' - \frac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}x'^2} \quad , \quad y^2 > 2\ s\ \tan(\alpha)\ x'$$

Für welche $x' = f$ **ist** $y = s\tan(\alpha)$ **?**

$$y^2 = 2\ s\tan(\alpha)\,x' - \frac{a}{\cos^2(\alpha)}x'^2 \quad , \quad a := \cos^2(\alpha) - \sin^2(\theta) < 0$$

$$s^2\tan^2(\alpha) = 2\ s\tan(\alpha)\,f - \frac{a}{\cos^2(\alpha)}f^2$$

$$\frac{a}{\cos^2(\alpha)}f^2 - 2\ s\tan(\alpha)\,f + s^2\tan^2(\alpha) = 0$$

$$f_{1/2} = \frac{2s\tan(\alpha) - \sqrt{4s^2\tan^2(\alpha) - 4\frac{a\,s^2\tan^2(\alpha)}{\cos^2(\alpha)}}}{2\ \frac{a}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{2s\tan(\alpha) \pm \sqrt{4s^2\tan^2(\alpha)\left(1 - \frac{a}{\cos^2(\alpha)}\right)}}{2\ \frac{a}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{2s\tan(\alpha) \pm 2s\tan(\alpha)\sqrt{\dfrac{\cos^2(\alpha)-a}{\cos^2(\alpha)}}}{2\,\dfrac{a}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{2s\tan(\alpha) \pm 2s\tan(\alpha)\sqrt{\dfrac{\cos^2(\alpha)-\left(\cos^2(\alpha)-\sin^2(\theta)\right)}{\cos^2(\alpha)}}}{2\,\dfrac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{2s\tan(\alpha) \pm 2s\tan(\alpha)\sqrt{\dfrac{\sin^2(\theta)}{\cos^2(\alpha)}}}{2\,\dfrac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{2s\tan(\alpha) \pm 2s\tan(\alpha)\dfrac{\sin(\theta)}{\cos(\alpha)}}{2\,\dfrac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{s\tan(\alpha) \pm s\tan(\alpha)\dfrac{\sin(\theta)}{\cos(\alpha)}}{\dfrac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}}$$

$$f_{1/2} = \frac{s\tan(\alpha)\left(1 \pm \dfrac{\sin(\theta)}{\cos(\alpha)}\right)}{1-\dfrac{\sin^2(\theta)}{\cos^2(\alpha)}}$$

$$f_1 = \frac{s\tan(\alpha)}{1-\dfrac{\sin(\theta)}{\cos(\alpha)}} \qquad f_2 = \frac{s\tan(\alpha)}{1+\dfrac{\sin(\theta)}{\cos(\alpha)}}$$

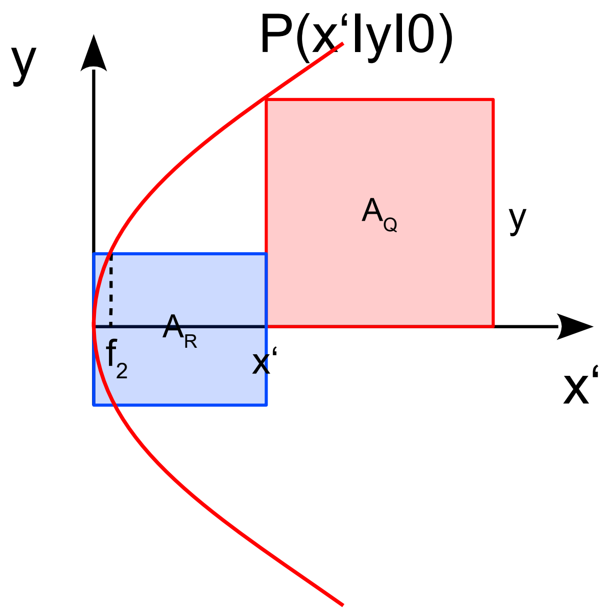
$$\underline{f_1 = \frac{s\sin(\alpha)}{\cos(\alpha)-\sin(\theta)}} \qquad \underline{f_2 = \frac{s\sin(\alpha)}{\cos(\alpha)+\sin(\theta)}}$$

Damit hat die Kurve die Punkte

$$\underline{F_1\left(\frac{s\sin(\alpha)}{\cos(\alpha)-\sin(\theta)} \;\; \mathrm{I} \;\; s\tan(\alpha)\right)} \qquad \underline{F_2\left(\frac{s\sin(\alpha)}{\cos(\alpha)+\sin(\theta)} \;\; \mathrm{I} \;\; s\tan(\alpha)\right)}$$

Geometrische Interpretation der Gleichung

$$\boxed{y^2 = 2\ s\tan(\alpha)\ x' - \frac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}x'^2} \quad , \quad y^2 > 2\ s\tan(\alpha)\ x'$$

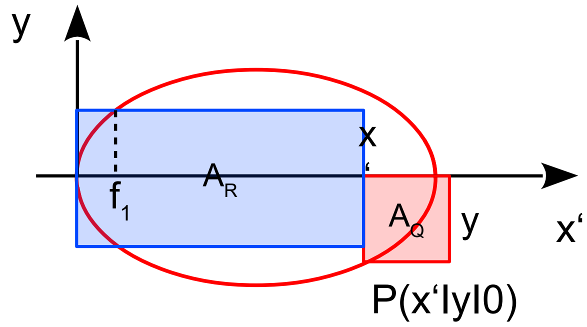


$$y^2 > 2\ s\tan(\alpha)\ x'$$

$$\underline{A_Q} > \underline{A_R}$$

Da der Flächeninhalt des Quadrats größer als der des Rechtecks ist, nennt man die Kurve **Hyperbel**, von gr. hyperballein = υπερβαλλειν = übertreffen .

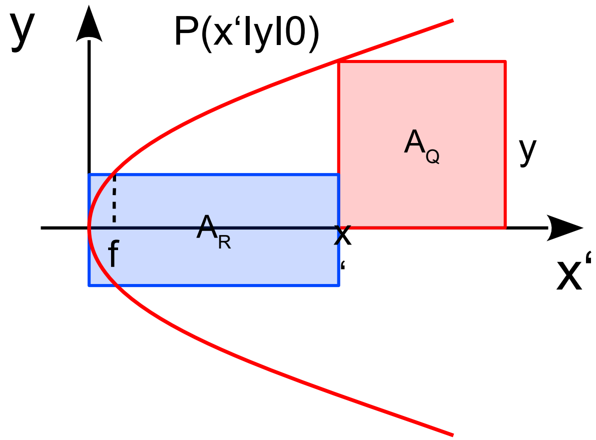
Ellipsengleichung in x‘, y



$$\boxed{y^2 = 2\ s \tan(\alpha)\ x' - \frac{\cos^2(\alpha) - \sin^2(\theta)}{\cos^2(\alpha)} x'^2}$$

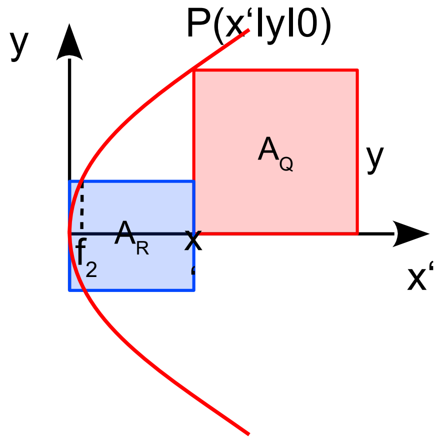
$$y^2 < 2\ s \tan(\alpha)\ x'$$

Parabelgleichung in x‘, y



$$\boxed{y^2 = 2\ \frac{s \sin(\theta) \cos(\theta)}{\cos^2(\alpha)}\ x'}$$

Hyperbelgleichung in x‘, y



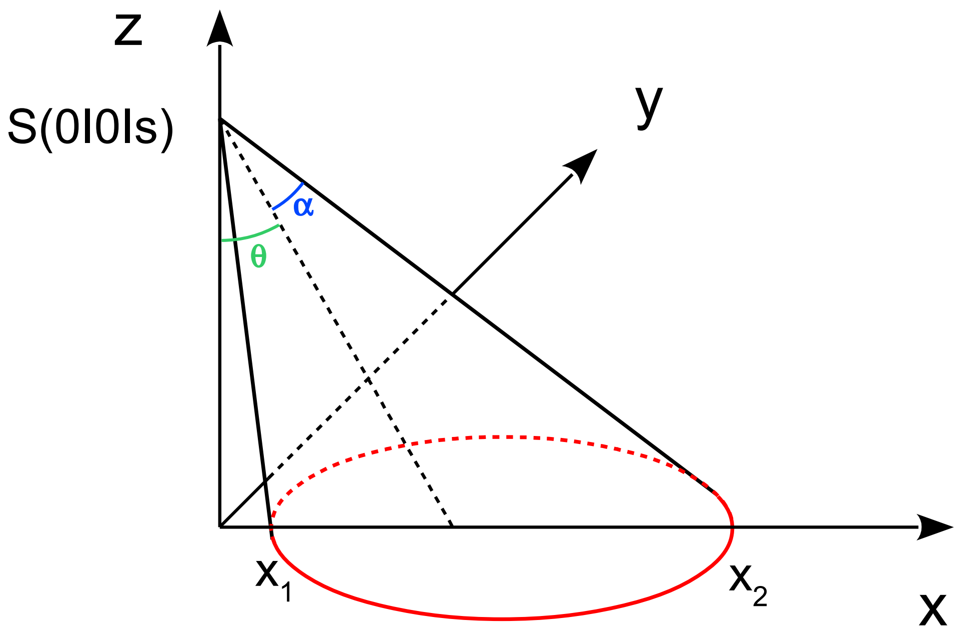
$$\boxed{y^2 = 2\ s \tan(\alpha)\ x' - \frac{\cos^2(\alpha) - \sin^2(\theta)}{\cos^2(\alpha)} x'^2}$$

$$y^2 > 2\ s \tan(\alpha)\ x'$$

Gleichungen der Kegelschnitte (minimalistisch)

Ein Kegel mit Spitze $S(0 \mid 0 \mid s)$ und halbem Öffnungswinkel α sei bezüglich der Symmetrieachse um den Winkel θ mit $0 \leq \theta < 90°+\alpha$ in Richtung x-Achse geneigt.

Gesucht ist die Gleichung der Schnittkurve des Kegels mit der xy-Ebene.



Vorgehensweise :

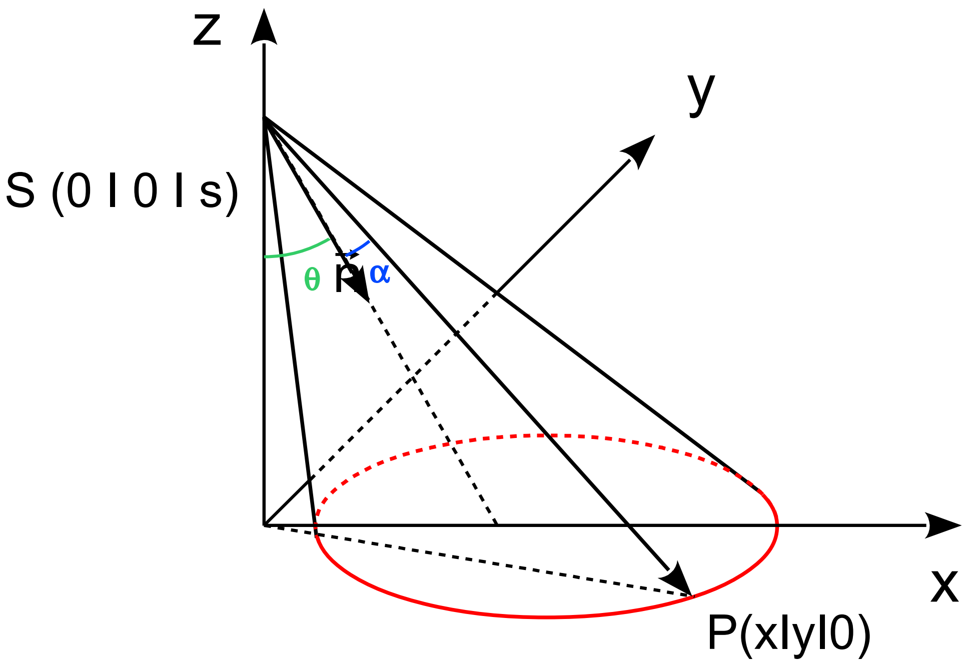
(1) Gleichung der Schnittkurve des Kegels mit der xy-Ebene

(2) Schnittpunkte der Kurve mit der x-Achse : $x_1 < x_2$

(3) Koordinatentransformation $x' = x - x_1 \quad , \quad y' = y$

(4) Darstellung der Schnittkurve in den neuen Koordinaten x‘ , y

(1) Gleichung der Schnittkurve des Kegels mit der xy-Ebene



$$\overrightarrow{SP} = \begin{pmatrix} x \\ y \\ -s \end{pmatrix} \qquad \vec{n} = \begin{pmatrix} \sin(\theta) \\ 0 \\ -\cos(\theta) \end{pmatrix}$$

$$|\overrightarrow{SP}| = \sqrt{x^2+y^2+s^2}$$

$$\overrightarrow{SP} = \begin{pmatrix} x \\ y \\ -s \end{pmatrix} \quad , \quad |\overrightarrow{SP}| = \sqrt{x^2+y^2+s^2} \quad , \quad \vec{n} = \begin{pmatrix} \sin(\theta) \\ 0 \\ -\cos(\theta) \end{pmatrix} \quad , \quad |\vec{n}| = 1$$

$$|\overrightarrow{SP}| \cdot \cos(\alpha) = \overrightarrow{SP} \cdot \vec{n}$$

$$\sqrt{x^2+y^2+s^2} \cdot \cos(\alpha) = \sin(\theta)\, x + s\cos(\theta)$$

$$\left(x^2+y^2+s^2\right) \cdot \cos^2(\alpha) = \sin^2(\theta)\, x^2 + 2s\sin(\theta)\cos(\theta)\, x + s^2\cos^2(\theta)$$

$$\cos^2(\alpha)x^2+\cos^2(\alpha)y^2+s^2\cos^2(\alpha) - \sin^2(\theta)x^2 - 2s\sin(\theta)\cos(\theta) - s^2\cos^2(\theta) = 0$$

$$\boxed{\left(\cos^2(\alpha)-\sin^2(\theta)\right)x^2 - 2s\sin(\theta)\cos(\theta)x + \cos^2(\alpha)y^2 + s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) = 0}$$

Gleichung der Schnittkurve des Kegels mit der xy-Ebene

(2) Schnittpunkte der Kurve mit der x-Achse : $x_1 < x_2$

$$\left(\cos^2(\alpha)-\sin^2(\theta)\right)x^2 - 2s\,\sin(\theta)\cos(\theta)\,x + \cos^2(\alpha)\,y^2 + s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) = 0$$

$$y = 0$$

$$\left(\cos^2(\alpha)-\sin^2(\theta)\right)x^2 - 2s\,\sin(\theta)\cos(\theta)\,x + s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right) = 0$$

Setze :

$$\underline{a := \cos^2(\alpha)-\sin^2(\theta)} \qquad \underline{b := -2s\,\sin(\theta)\cos(\theta)} \qquad \underline{c := s^2\left(\cos^2(\alpha)-\cos^2(\theta)\right)}$$

$$a\,x^2 + bx + c = 0$$

$$\underline{x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}} \quad < \quad \underline{x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}}$$

(3) Koordinatentransformation $x' = x - x_1 \quad , \quad y' = y$

$$x = x' + x_1$$

$$\left(\cos^2(\alpha) - \sin^2(\theta)\right) x^2 - 2s \sin(\theta)\cos(\theta) x + \cos^2(\alpha) y^2 + s^2\left(\cos^2(\alpha) - \cos^2(\theta)\right) = 0$$

$$a x^2 + bx + \cos^2(\alpha) y^2 + c = 0$$

$$a\left(x'+x_1\right)^2 + b\left(x'+x_1\right) + \cos^2(\alpha) y^2 + c = 0$$

$$a x'^2 + 2a x_1 x' + a x_1^2 + b x' + b x_1 + \cos^2(\alpha) y^2 + c = 0$$

$$a x'^2 + 2a x_1 x' + b x' + \cos^2(\alpha) y^2 + \underbrace{a x_1^2 + b x_1 + c}_{=0} = 0$$

$$a x'^2 + \left(2a x_1 + b\right) x' + \cos^2(\alpha) y^2 = 0$$

$$\frac{a}{\cos^2(\alpha)} \; x'^2 \; + \; \frac{(2ax_1 + b)}{\cos^2(\alpha)} \; x' \; + \; y^2 \; = \; 0$$

Nebenrechnung :

$$\frac{(2ax_1 + b)}{\cos^2(\alpha)} \; = \; \frac{-\sqrt{b^2 - 4ac}}{\cos^2(\alpha)}$$

$$\frac{(2ax_1 + b)}{\cos^2(\alpha)} \; = \; \frac{-\sqrt{b^2 - 4ac}}{\cos^2(\alpha)}$$

$\underline{a := \cos^2(\alpha) - \sin^2(\theta)}$ $\qquad$ $\underline{b := -2s\sin(\theta)\cos(\theta)}$ $\qquad$ $\underline{c := s^2\left(\cos^2(\alpha) - \cos^2(\theta)\right)}$

$$\frac{(2ax_1 + b)}{\cos^2(\alpha)} \; = \; \frac{-\sqrt{4s^2\sin^2(\theta)\cos^2(\theta) - 4\left(\cos^2(\alpha) - \sin^2(\theta)\right)s^2\left(\cos^2(\alpha) - \cos^2(\theta)\right)}}{\cos^2(\alpha)}$$

$$\frac{\left(2ax_1 + b\right)}{\cos^2(\alpha)} = -\sqrt{\frac{4s^2\sin^2(\theta)\cos^2(\theta)}{\cos^2(\alpha)\cos^2(\alpha)} - \frac{4s^2\left(\cos^2(\alpha)-\sin^2(\theta)\right)}{\cos^2(\alpha)}\frac{\left(\cos^2(\alpha)-\cos^2(\theta)\right)}{\cos^2(\alpha)}}$$

$$\frac{\left(2ax_1 + b\right)}{\cos^2(\alpha)} = -\sqrt{\frac{4s^2\sin^2(\theta)\cos^2(\theta)}{\cos^2(\alpha)\cos^2(\alpha)} - 4s^2\left(1-\frac{\sin^2(\theta)}{\cos^2(\alpha)}\right)\left(1-\frac{\cos^2(\theta)}{\cos^2(\alpha)}\right)}$$

$$\frac{\left(2ax_1 + b\right)}{\cos^2(\alpha)} = -\sqrt{\frac{4s^2\sin^2(\theta)\cos^2(\theta)}{\cos^2(\alpha)\cos^2(\alpha)} - 4s^2\left(1 - \frac{\cos^2(\theta)}{\cos^2(\alpha)} - \frac{\sin^2(\theta)}{\cos^2(\alpha)} + \frac{\sin^2(\theta)\cos^2(\theta)}{\cos^2(\alpha)\cos^2(\alpha)}\right)}$$

$$\frac{\left(2ax_1 + b\right)}{\cos^2(\alpha)} = -\sqrt{4s^2\left(\frac{\cos^2(\theta)}{\cos^2(\alpha)} + \frac{\sin^2(\theta)}{\cos^2(\alpha)} - 1\right)}$$

$$\frac{\left(2ax_1 + b\right)}{\cos^2(\alpha)} = -\sqrt{4s^2\left(\frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\alpha)} - 1\right)}$$

$$\frac{\left(2ax_1 + b\right)}{\cos^2(\alpha)} = -\sqrt{4s^2\left(\frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\alpha)} - 1\right)}$$

$$\frac{(2ax_1 + b)}{\cos^2(\alpha)} = -\sqrt{4s^2\left(\frac{1}{\cos^2(\alpha)} - 1\right)}$$

$$\frac{(2ax_1 + b)}{\cos^2(\alpha)} = -\sqrt{4s^2\left(\frac{1 - \cos^2(\alpha)}{\cos^2(\alpha)}\right)}$$

$$\frac{(2ax_1 + b)}{\cos^2(\alpha)} = -\sqrt{4s^2\left(\frac{\sin^2(\alpha)}{\cos^2(\alpha)}\right)}$$

$$\frac{(2ax_1 + b)}{\cos^2(\alpha)} = -\sqrt{4s^2\tan^2(\alpha)}$$

$$\underline{\frac{(2ax_1 + b)}{\cos^2(\alpha)} = -2s\tan(\alpha)}$$

$$\underline{a := \cos^2(\alpha) - \sin^2(\theta)}$$

Weiter mit der Schnittgleichung :

$$\frac{a}{\cos^2(\alpha)}\, x'^2 + \frac{(2ax_1 + b)}{\cos^2(\alpha)}\, x' + y^2 = 0$$

$$\frac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)}\, x'^2 - 2s\tan(\alpha)\,x' + y^2 = 0$$

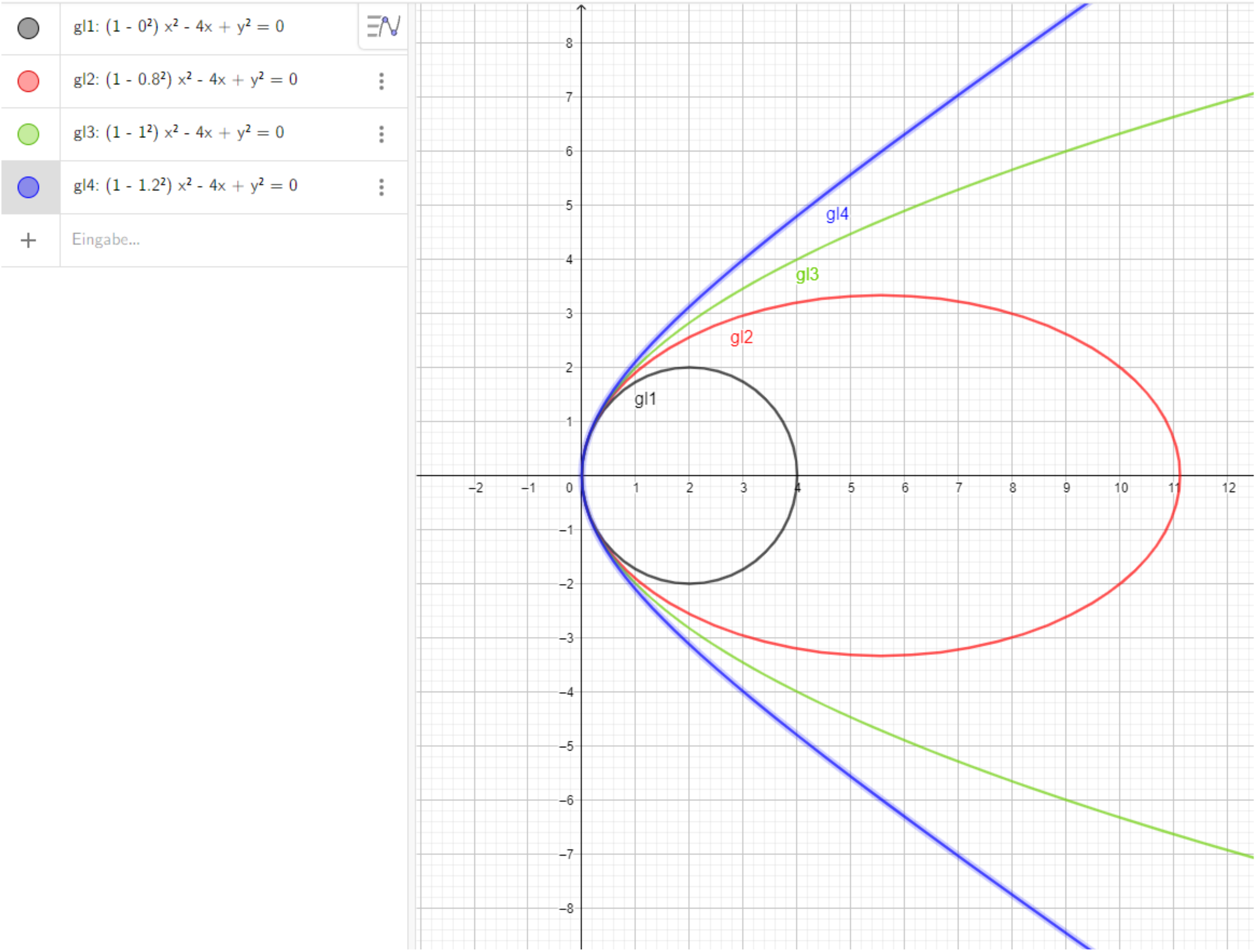
Setze :

$\epsilon := \dfrac{\sin(\theta)}{\cos(\alpha)}$ Numerische Exzentrizität $\Rightarrow$ $\dfrac{\cos^2(\alpha)-\sin^2(\theta)}{\cos^2(\alpha)} = 1 - \epsilon^2$

$$\boxed{\left(1-\epsilon^2\right) x'^2 - 2s\tan(\alpha)\, x' + y^2 = 0}$$

Gleichung der Schnittkurve des Kegels mit der x‘y-Ebene parametrisiert durch die numerische Exzentrizität ϵ , der Spitzenkoordinate s und den halben Kegelwinkel α .

Je nach Größe von ϵ ergeben sich unterschiedliche Kurven :



$\varepsilon = 0$	:	Kreis
$0 < \varepsilon < 1$	:	Ellipse
$\varepsilon = 1$	:	Parabel
$1 < \varepsilon$	:	Hyperbel

Durch die Vorgabe des halben Öffnungswinkels α_1 und der Kegelspitze $S_1(0|0|s_1)$ sowie der Wahl des Neigungswinkels θ_1 kann man unendlich viele Kegelschnitte erzeugen :

$$\left(1-\epsilon_1^2\right)\ x^2\ -\ 2s_1\tan(\alpha_1)\ x\ +\ y^2\ =\ 0 \quad \text{mit} \quad \epsilon_1 = \frac{\sin(\theta_1)}{\cos(\alpha_1)}$$

Kann man bei einem zweiten Kegel mit dem halben Öffnungswinkel α_2 eine Spitze $S_2(0|0|s_2)$ und einen Neigungswinkel θ_2 angeben, der den gleichen Kegelschnitt erzeugt ?

$$\left(1-\epsilon_2^2\right)\ x^2\ -\ 2s_2\tan(\alpha_2)\ x\ +\ y^2\ =\ 0$$

mit $\epsilon_2 = \dfrac{\sin(\theta_2)}{\cos(\alpha_2)}$

und $2s_2\tan(\alpha_2) = 2s_1\tan(\alpha_1) \quad , \quad \epsilon_2 = \epsilon_1$

Lösung :

$$2s_2 \tan(\alpha_2) = 2s_1 \tan(\alpha_1)$$

$$s_2 = s_1 \frac{\tan(\alpha_1)}{\tan(\alpha_2)}$$

$$\epsilon_2 = \epsilon_1$$

$$\frac{\sin(\theta_2)}{\cos(\alpha_2)} = \frac{\sin(\theta_1)}{\cos(\alpha_1)}$$

$$\sin(\theta_2) = \sin(\theta_1) \frac{\cos(\alpha_2)}{\cos(\alpha_1)}$$

$$\theta_2 = \sin^{-1}\left(\sin(\theta_1) \frac{\cos(\alpha_2)}{\cos(\alpha_1)}\right)$$