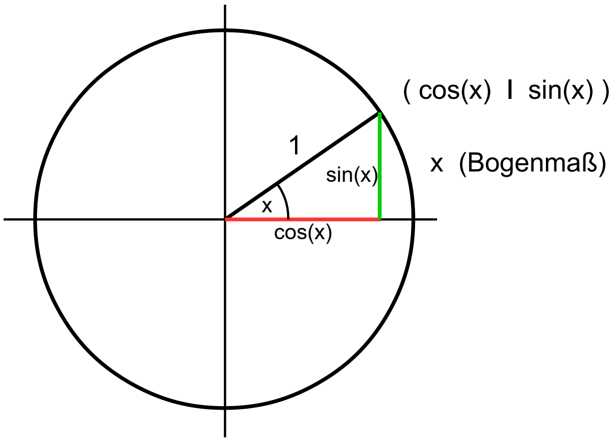
Die Trigonometrischen Funktionen y = sin (x)   ,   y = cos(x)

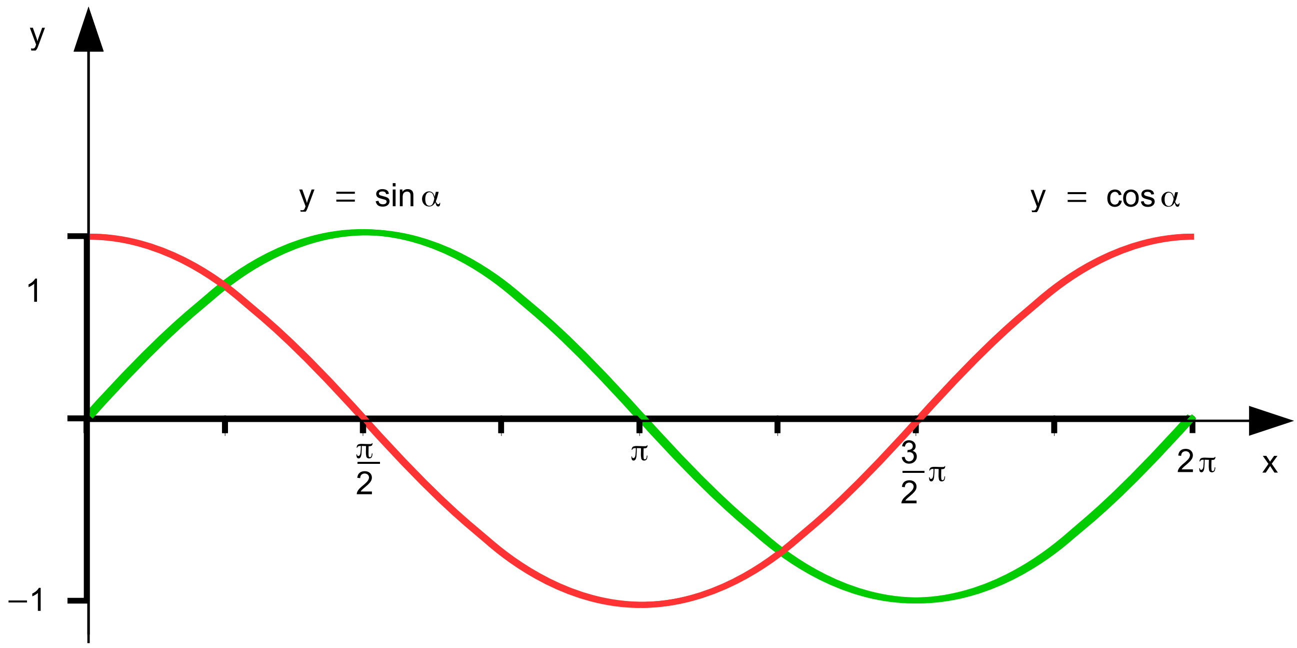
Trigonometrische Funktionen am Einheitskreis :



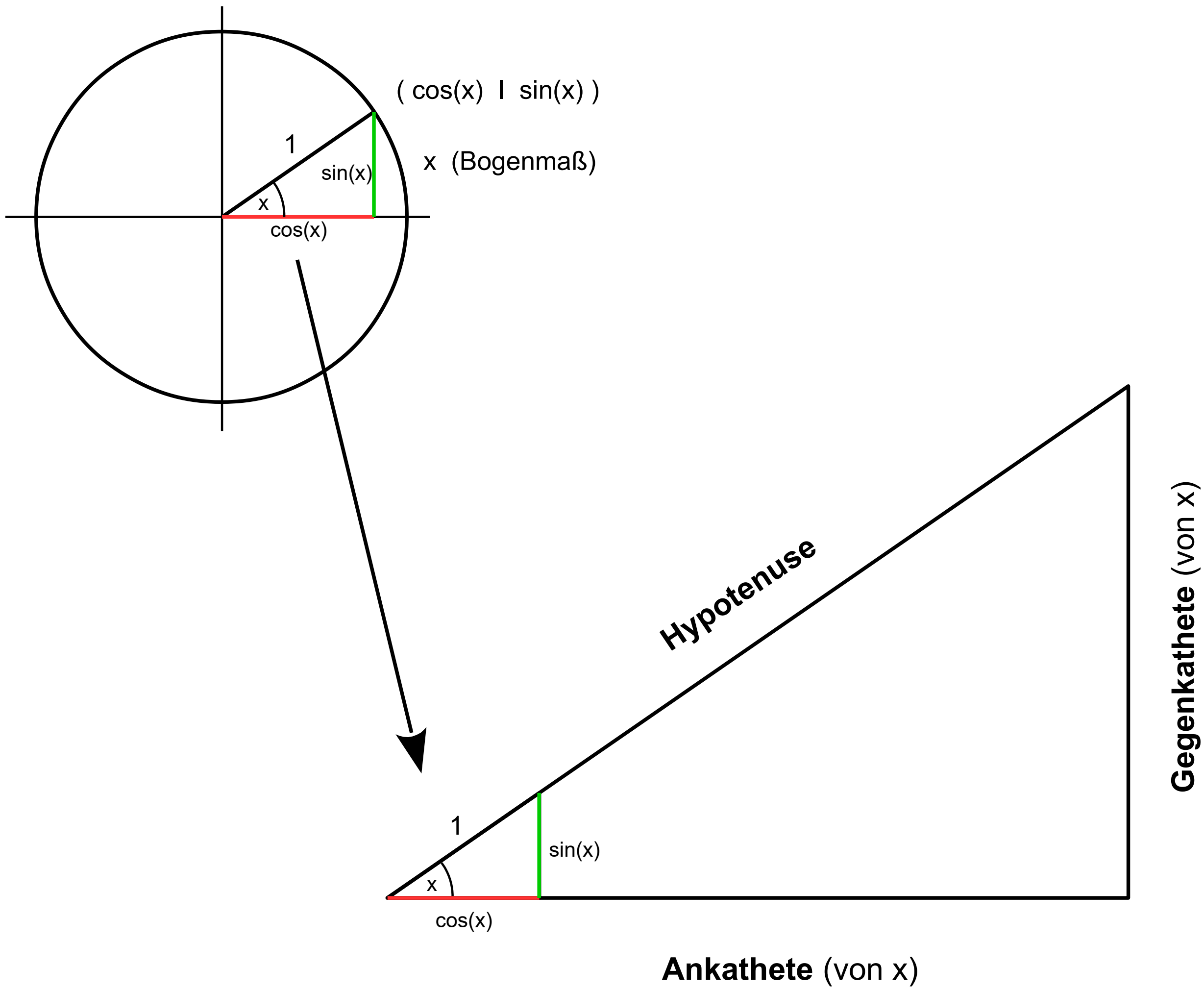
Satz des Pythagoras für trigonometrische Funktionen :

$$\boxed{\sin^2(x) + \cos^2(x) = 1}$$

Schaubilder



Trigonometrische Gleichungen am rechtwinkligen Dreieck

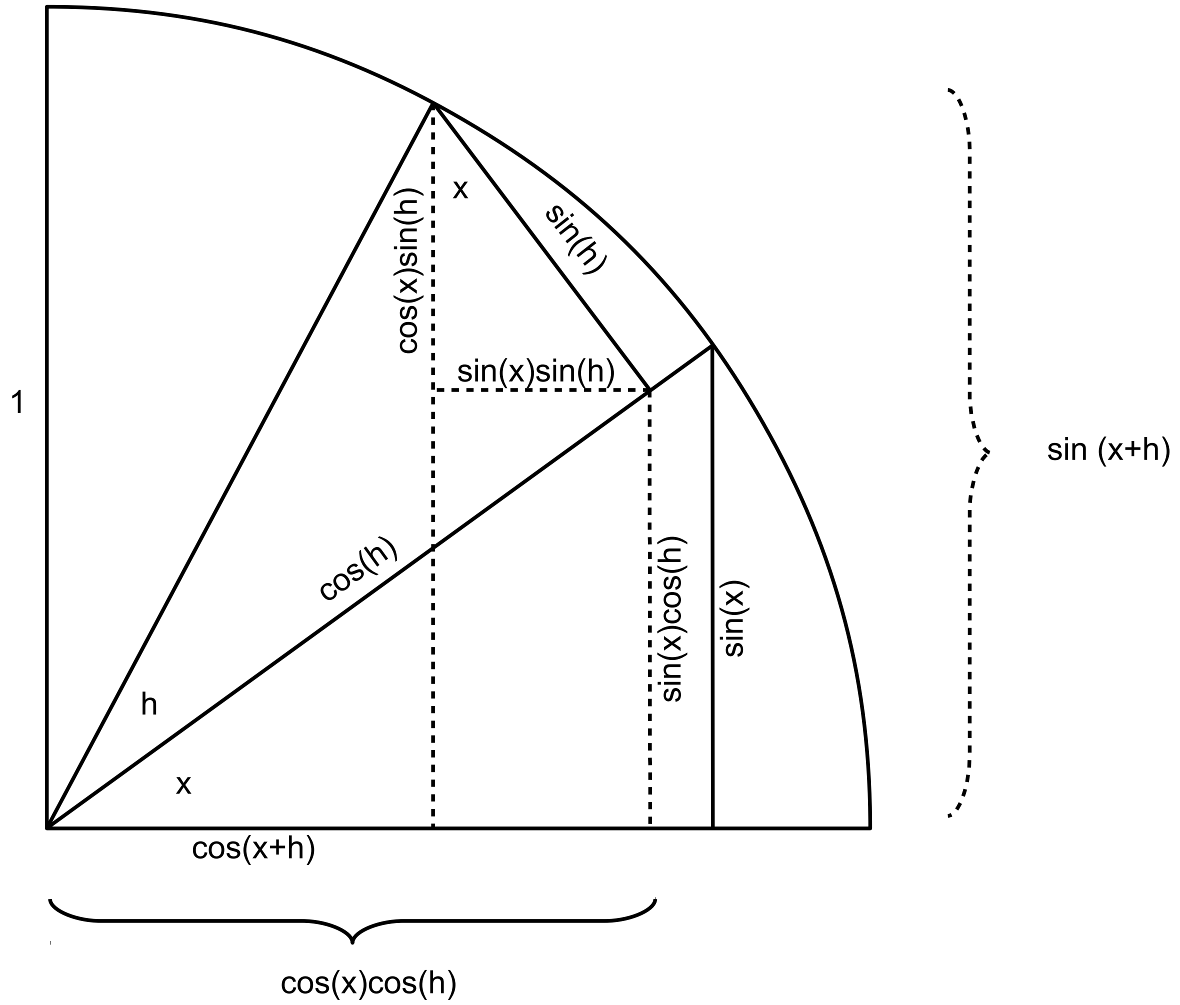


2. Strahlensatz

$$\frac{\sin(x)}{1} = \frac{\text{Gegenkathete}}{\text{Hypotenuse}} \quad \Rightarrow \quad \boxed{\sin(x) = \frac{\text{Gegenkathete}}{\text{Hypotenuse}}}$$

$$\frac{\cos(x)}{1} = \frac{\text{Ankathete}}{\text{Hypotenuse}} \quad \Rightarrow \quad \boxed{\cos(x) = \frac{\text{Ankathete}}{\text{Hypotenuse}}}$$

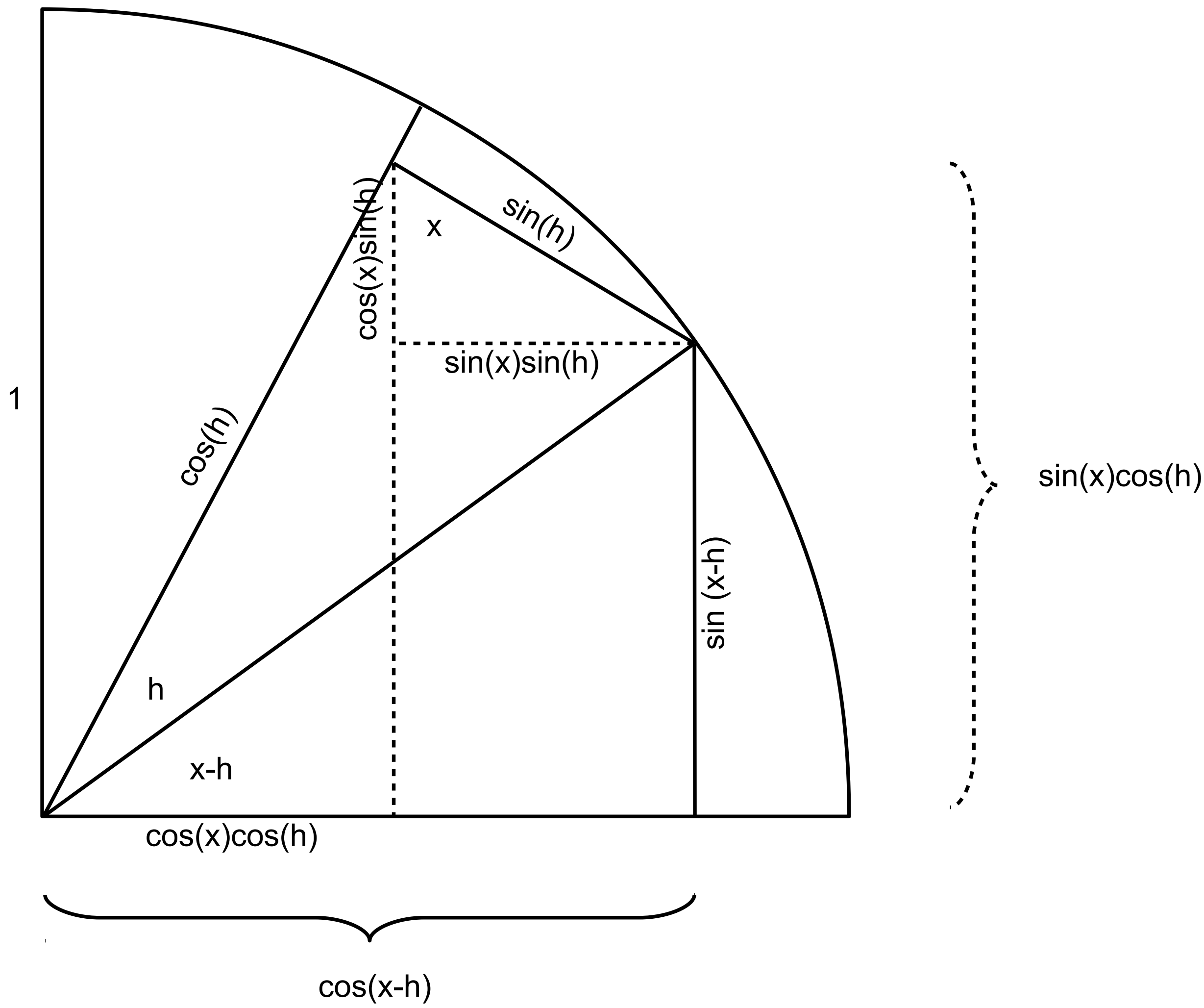
Additionstheoreme für y = sin(x) , y = cos(x)



$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

$$\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h)$$

Subtraktionstheoreme für y = sin(x) , y = cos(x)



$$\sin(x-h) = \sin(x)\cos(h) - \cos(x)\sin(h)$$

$$\cos(x-h) = \cos(x)\cos(h) + \sin(x)\sin(h)$$

Ableitung der Trigonometrischen Funktionen y = sin(x) , y = cos(x)

$$\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$\sin'(x) = \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$\sin'(x) = \lim_{h\to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}$$

$$\sin'(x) = \lim_{h\to 0} \sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h}$$

Wegen $\lim\limits_{h\to 0} \dfrac{\cos(h) - 1}{h} = 0$ und $\lim\limits_{h\to 0} \dfrac{\sin(h)}{h} = 1$, was auf den folgenden Seiten gezeigt wird, folgt :

$$\boxed{\sin'(x) = \cos(x)}$$

$$\cos'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$\cos'(x) = \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

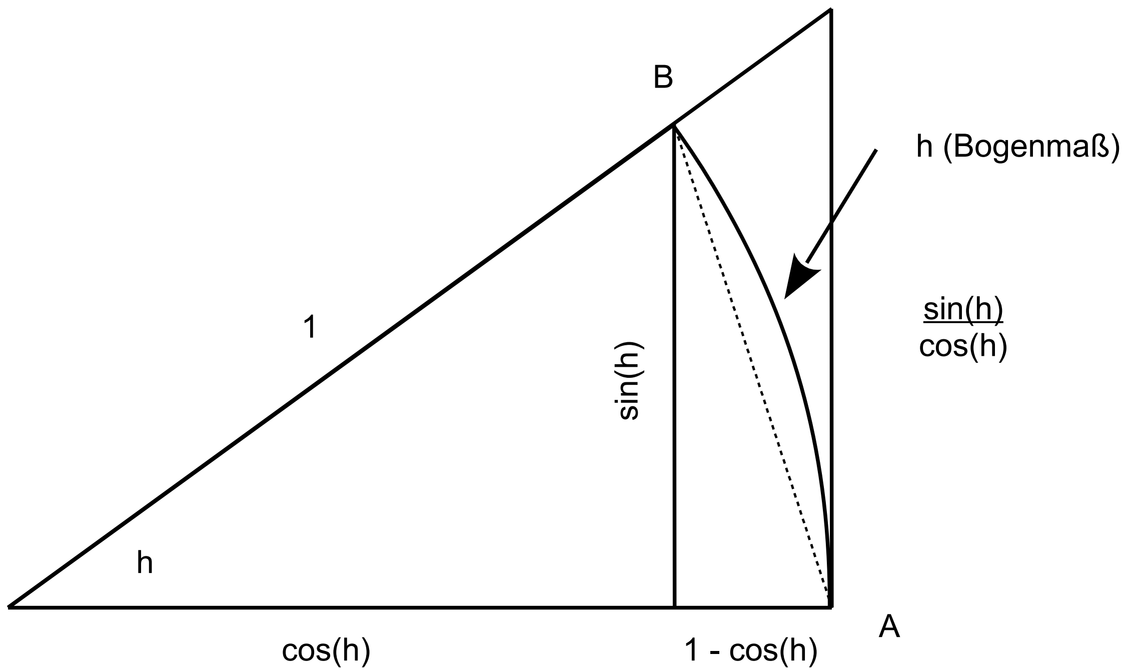
$$\cos'(x) = \lim_{h\to 0} \frac{\cos(x)(\cos(h) - 1) - \sin(x)\sin(h)}{h}$$

$$\cos'(x) = \lim_{h\to 0} \cos(x) \cdot \frac{(\cos(h) - 1)}{h} - \sin(x) \cdot \frac{\sin(h)}{h}$$

Wiederum wegen $\lim\limits_{h\to 0} \dfrac{\cos(h) - 1}{h} = 0$ und $\lim\limits_{h\to 0} \dfrac{\sin(h)}{h} = 1$ folgt :

$$\boxed{\cos'(x) = -\sin(x)}$$

Grenzwerte und Abschätzungen der Trigonometrischen Funktionen



Simple Grenzwerte :

$$\sin(h) < |AB| < h \quad \Rightarrow \quad \boxed{\lim_{h\to 0} \sin(h) = 0}$$

$$1-\cos(h) < |AB| < h \quad \Rightarrow \quad \lim_{h\to 0} 1-\cos(h) = 0$$

$$\Rightarrow \quad \boxed{\lim_{h\to 0} \cos(h) = 1}$$

Grenzwert $\lim\limits_{h\to 0} \frac{\sin(h)}{h}$:

Flächenbetrachtung :

$$\frac{1}{2} \cdot 1 \cdot \sin(h) < \frac{1}{2} \cdot h \cdot 1 < \frac{1}{2} \cdot 1 \cdot \frac{\sin(h)}{\cos(h)}$$

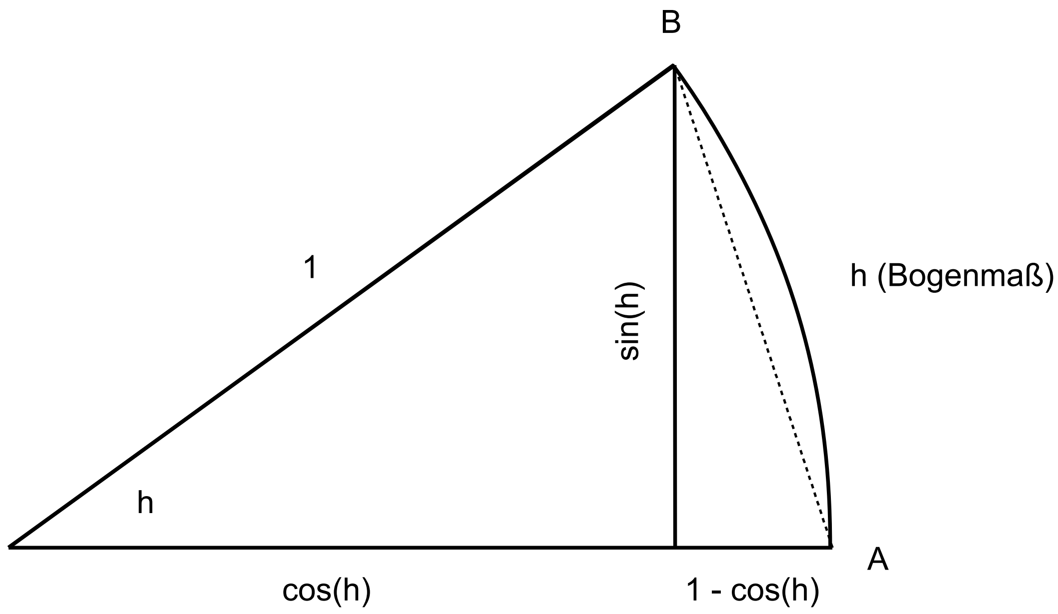
$$\sin(h) < h < \frac{\sin(h)}{\cos(h)}$$

$$1 < \frac{h}{\sin(h)} < \frac{1}{\cos(h)} \quad , \quad \text{falls} \quad h > 0$$

$$1 > \frac{h}{\sin(h)} > \frac{1}{\cos(h)} \quad , \quad \text{falls} \quad h < 0$$

$$\lim_{h\to 0} \frac{h}{\sin(h)} = 1 \quad \Rightarrow \quad \boxed{\lim_{h\to 0} \frac{\sin(h)}{h} = 1}$$

Grenzwert $\lim\limits_{h \to 0} \dfrac{\cos(h) - 1}{h}$:



$$|AB| < h$$

$$|AB|^2 = (1 - \cos(h))^2 + \sin^2(h)$$

$$|AB|^2 = 1 - 2\cos(h) + \cos^2(h) + \sin^2(h)$$

$$|AB|^2 = 1 - 2\cos(h) + 1$$

$$|AB|^2 = 2 - 2\cos(h)$$

$$|AB|^2 = 2(1 - \cos(h))$$

$$1 - \cos(h) = \frac{|AB|^2}{2}$$

$$1 - \cos(h) < \frac{h^2}{2}$$

$$\frac{1 - \cos(h)}{h} < \frac{h}{2}$$

$$\lim_{h \to 0} \frac{1 - \cos(h)}{h} = 0 \qquad \Rightarrow \qquad \boxed{\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0}$$

Grenzwert $\lim\limits_{h\to 0} \dfrac{\cos(h) - 1}{h}$, **1. Alternative** :

$$\lim_{h\to 0} \frac{\left(\cos(h) - 1\right)}{h} = \lim_{h\to 0} \frac{\cos\left(\frac{h}{2}+\frac{h}{2}\right) - 1}{h}$$

$$\lim_{h\to 0} \frac{\left(\cos(h) - 1\right)}{h} = \lim_{h\to 0} \frac{\cos^2\left(\frac{h}{2}\right) - \sin^2\left(\frac{h}{2}\right) - 1}{h}$$

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = \lim_{h\to 0} \frac{1-\sin^2\left(\frac{h}{2}\right) - \sin^2\left(\frac{h}{2}\right) - 1}{h}$$

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = \lim_{h\to 0} \frac{-2\sin^2\left(\frac{h}{2}\right)}{h}$$

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = \lim_{h\to 0} \frac{-2\sin^2\left(\frac{h}{2}\right)}{2\frac{h}{2}}$$

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = \lim_{h\to 0} \frac{-\sin^2\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = \lim_{h\to 0} -\sin\left(\frac{h}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

Wegen $\lim\limits_{h\to 0} \sin(h) = 0$ und $\lim\limits_{h\to 0} \dfrac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = 1$ folgt :

$$\boxed{\lim_{h\to 0} \frac{\left(\cos(h) - 1\right)}{h} = 0}$$

Grenzwert $\lim\limits_{h \to 0} \frac{\cos(h) - 1}{h}$, **2. Aternative** :

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = \lim_{h \to 0} \frac{(\cos(h) - 1)\cdot(\cos(h) + 1)}{h\cdot(\cos(h) + 1)}$$

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = \lim_{h \to 0} \frac{\cos^2(h) - 1}{h\cdot(\cos(h) + 1)}$$

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = \lim_{h \to 0} \frac{-\sin^2(h)}{h\cdot(\cos(h) + 1)}$$

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = \lim_{h \to 0} \frac{-\sin(h)}{h} \cdot \frac{\sin(h)}{\cos(h) + 1}$$

Wegen $\lim\limits_{h \to 0} \frac{-\sin(h)}{h} = -1$ und $\lim\limits_{h \to 0} \frac{\sin(h)}{\cos(h) + 1} = \frac{0}{1 + 1} = 0$ folgt :

$$\boxed{\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0}$$

Ableitung der Trigonometrischen Funktionen y = sin(x) , y = cos(x) , y = tan(x)

$$\boxed{\sin'(x) = \cos(x)}$$

$$\boxed{\cos'(x) = -\sin(x)}$$

$$\tan(x) := \frac{\sin(x)}{\cos(x)} \quad \Leftrightarrow \quad \tan(x) := \frac{\frac{\text{Gegenkathete}}{\text{Hypotenuse}}}{\frac{\text{Ankathete}}{\text{Hypotenuse}}} = \frac{\text{Gegenkathete}}{\text{Ankathete}}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\tan'(x) = \frac{\sin'(x)\cdot\cos(x) - \sin(x)\cdot\cos'(x)}{\cos^2(x)}$$

$$\tan'(x) = \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)}$$

$$\tan'(x) = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$\boxed{\tan'(x) = \frac{1}{\cos^2(x)}}$$