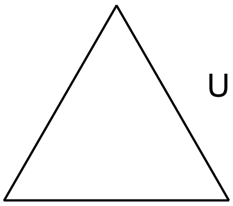
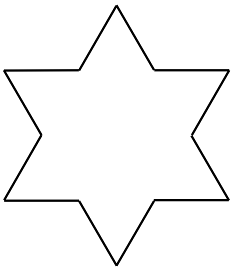
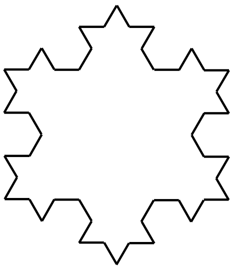
Schneeflockenkurve, Kochkurve (Helge von Koch, 1870 - 1924)



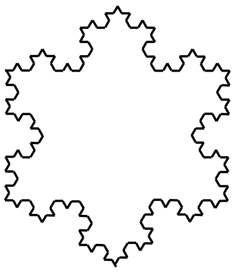
$$U_1 = U$$



$$U_2 = \frac{4}{3}U$$



$$U_3 = \left(\frac{4}{3}\right)^2 U$$



$$U_4 = \left(\frac{4}{3}\right)^3 U$$

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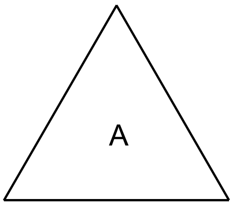
.
.

$$U_n = \left(\frac{4}{3}\right)^{n-1} U$$

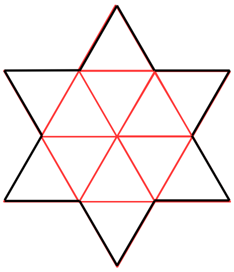
Schneeflockenkurve Sfk

$$U(\mathrm{Sfk}) = \lim_{n\to\infty} U_n = \lim_{n\to\infty} \left(\frac{4}{3}\right)^{n-1} U = \infty$$

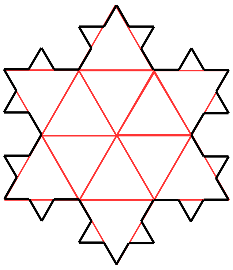
Schneeflockenkurve, Kochkurve (Helge von Koch, 1870 - 1924)



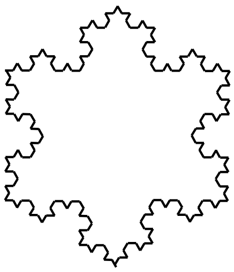
$$A_1 = A$$



$$A_2 = A + 3\frac{A}{9}$$



$$A_3 = A + 3\frac{A}{9} + 3\cdot 4\frac{A}{9^2}$$



$$A_4 = A + 3\frac{A}{9} + 3\cdot 4\frac{A}{9^2} + 3\cdot 4^2\frac{A}{9^3}$$

$$A_5 = A + 3\frac{A}{9} + 3\cdot 4\frac{A}{9^2} + 3\cdot 4^2\frac{A}{9^3} + 3\cdot 4^3\frac{A}{9^4}$$

.
.
.

$$A_6 = A + 3\frac{A}{9} + 3\cdot 4\frac{A}{9^2} + 3\cdot 4^2\frac{A}{9^3} + 3\cdot 4^3\frac{A}{9^4} + 3\cdot 4^4\frac{A}{9^5}$$

$$A_7 = A + 3\frac{A}{9} + 3\cdot 4\frac{A}{9^2} + 3\cdot 4^2\frac{A}{9^3} + 3\cdot 4^3\frac{A}{9^4} + 3\cdot 4^4\frac{A}{9^5} + 3\cdot 4^5\frac{A}{9^6}$$

$\downarrow$

Schneeflockenkurve Sfk $\quad A(\mathrm{Sfk}) = ?$

$$A_7 = A + 3\frac{A}{9} + 3\cdot 4\frac{A}{9^2} + 3\cdot 4^2\frac{A}{9^3} + 3\cdot 4^3\frac{A}{9^4} + 3\cdot 4^4\frac{A}{9^5} + 3\cdot 4^5\frac{A}{9^6}$$

$$A_7 = A + 3\frac{A}{9}\cdot\left(1 + 4\frac{1}{9^1} + 4^2\frac{1}{9^2} + 4^3\frac{1}{9^3} + 4^4\frac{1}{9^4} + 4^5\frac{1}{9^5}\right)$$

$$A_7 = A + 3\frac{A}{9}\cdot\left(1 + \frac{4^1}{9^1} + \frac{4^2}{9^2} + \frac{4^3}{9^3} + \frac{4^4}{9^4} + \frac{4^5}{9^5}\right)$$

$$A_7 = A + 3\frac{A}{9}\cdot\left(1 + \left(\frac{4}{9}\right)^1 + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \left(\frac{4}{9}\right)^4 + \left(\frac{4}{9}\right)^5\right)$$

Die n-te Näherung der Fläche der Schneeflockenkurve ist im Wesentlichen eine geometrische Reihe mit dem Faktor $q = \frac{4}{9} < 1$ (vgl. S. 5)

$$A_n = A + 3\frac{A}{9}\cdot\left(1 + \left(\frac{4}{9}\right)^1 + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \left(\frac{4}{9}\right)^4 + \ldots + \left(\frac{4}{9}\right)^{n-2}\right)$$

$$A_n = A + 3\frac{A}{9}\cdot\left(1 + \left(\frac{4}{9}\right)^1 + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \left(\frac{4}{9}\right)^4 + \ldots + \left(\frac{4}{9}\right)^{n-2}\right)$$

Deshalb existiert der Grenzwert $\lim\limits_{n\to\infty} A_n$

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} A + 3\frac{A}{9}\cdot\left(1 + \left(\frac{4}{9}\right)^1 + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \left(\frac{4}{9}\right)^4 + \ldots + \left(\frac{4}{9}\right)^{n-2}\right)$$

$$A(Sfk) = \lim_{n\to\infty} A_n = A + 3\frac{A}{9}\cdot\lim_{n\to\infty}\left(1 + \left(\frac{4}{9}\right)^1 + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \left(\frac{4}{9}\right)^4 + \ldots + \left(\frac{4}{9}\right)^{n-2}\right)$$

$$A(Sfk) = \lim_{n\to\infty} A_n = A + 3\frac{A}{9}\cdot\frac{1}{1-\frac{4}{9}}$$

$$A(Sfk) = \lim_{n\to\infty} A_n = A + 3\frac{A}{9}\cdot\frac{9}{5}$$

$$A(Sfk) = \lim_{n\to\infty} A_n = A + \frac{3}{5}A$$

$$A(Sfk) = \lim_{n\to\infty} A_n = \frac{8}{5}A$$

Geometrische Folgen

$q \in \mathbb{R}_0^+ \quad , \quad a_n = q^n \quad , \quad n \in \mathbb{N}$

1. Fall : $q = 1$

$a_n = q^n = 1$ für alle $n \in \mathbb{N}$ und $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} q^n = \lim\limits_{n\to\infty} 1 = 1$

2. Fall : $q = 0$

$a_n = q^n = 0$ für alle $n \in \mathbb{N}$ und $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} q^n = \lim\limits_{n\to\infty} 0 = 0$

3. Fall : $q > 1$, etwa $q = 1+a$, $a > 0$

$a_n = q^n = (1+a)^n \geq 1 + na$ und $\lim\limits_{n\to\infty} q^n = \infty$, also **divergent** .

4. Fall : $0 < q < 1$, etwa $q = \frac{1}{\overline{q}}$, $\overline{q} > 1$, $\overline{q} = 1+a$, $a > 0$

$$0 < a_n = q^n = \left(\frac{1}{\overline{q}}\right)^n = \frac{1}{\overline{q}^{\,n}} = \frac{1}{(1+a)^{\,n}} \leq \frac{1}{1 + na} \quad \text{und} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} q^n = 0$$

Geometrische Folgen

$q \in \mathbb{R}^- - \{0\} \quad , \quad a_n = q^n \quad , \quad n \in \mathbb{N}$

1. Fall : $q = -1$

$$a_n = (-1)^n = \begin{Bmatrix} 1 & , & \text{falls } n \text{ gerade} \\ -1 & , & \text{falls } n \text{ ungerade} \end{Bmatrix} \text{ , also } \textbf{divergent}.$$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} q^n = \lim\limits_{n\to\infty} 1 = 1$

2. Fall : $q < -1$, etwa $q = -1-a = -(1+a)$, $a > 0$

$$a_n = (-(1+a))^n = \begin{Bmatrix} (1+a)^n & , & \text{falls } n \text{ gerade} \\ -(1+a)^n & , & \text{falls } n \text{ ungerade} \end{Bmatrix} \text{ , also } \textbf{divergent}.$$

3. Fall : $-1 < q < 0$, etwa $q = -\frac{1}{\overline{q}}$, $\overline{q} > 1$, $0 < \frac{1}{\overline{q}} < 1$

$$a_n = \left(-\frac{1}{\overline{q}}\right)^n = \begin{Bmatrix} \left(\frac{1}{\overline{q}}\right)^n & , & \text{falls } n \text{ gerade} \\ -\left(\frac{1}{\overline{q}}\right)^n & , & \text{falls } n \text{ ungerade} \end{Bmatrix} \text{ . also } \lim_{n\to\infty} a_n = 0$$

Zusammenfassung :

Für $q \in \mathbb{R}$ ist die **geometrische Folge** $a_n = q^n$, $n \in \mathbb{N}$ gilt :

$-1 < q < 1$: $\boxed{\lim_{n\to\infty} q^n = 0}$

$q = 1$: $\lim_{n\to\infty} q^n = \lim_{n\to\infty} 1 = 1$

In allen anderen Fällen ist die **geometrische Folge divergent**.

Geometrische Reihen

$q \in \mathbb{R}$, $q \neq 1$

$$S_n = 1 + q^1 + q^2 + q^3 + q^4 + \ldots + q^{n-1} + q^n \quad , \quad n \in \mathbb{N}$$

$$S_n = 1 + q\left(1 + q^1 + q^2 + q^3 + \ldots + q^{n-1}\right)$$

$$S_n = 1 + q\left(1 + q^1 + q^2 + q^3 + \ldots + q^{n-1} + q^n\right) - q^{n+1}$$

$$S_n = 1 + qS_n - q^{n+1}$$

$$S_n - q \cdot S_n = 1 - q^{n+1}$$

$$S_n \cdot (1 - q) = 1 - q^{n+1}$$

$$\boxed{S_n = \frac{1 - q^{n+1}}{1 - q}} \quad , \quad q \neq 1$$

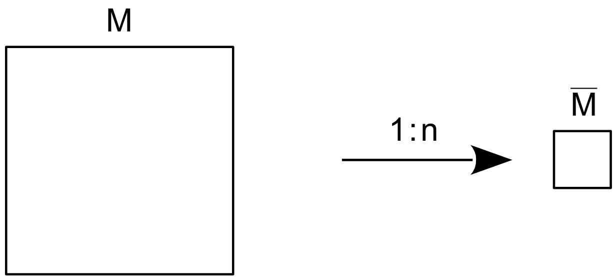
Falls $-1 < q < 1$ ist, folgt :

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1-q} \quad , \quad \text{wegen} \quad \lim_{n\to\infty} q^{n+1} = 0 \quad ,$$

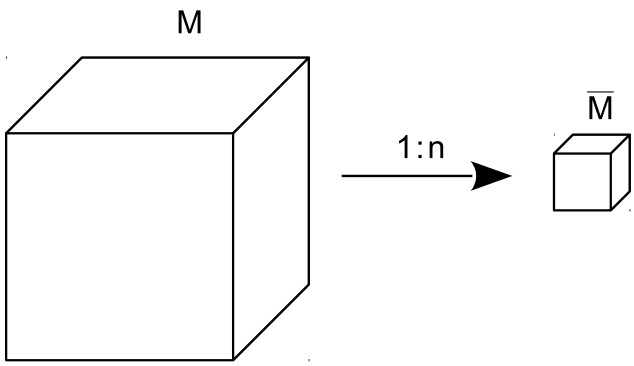
$$\boxed{\lim_{n\to\infty} S_n = \frac{1}{1-q}}$$

Dimensionsbetrachtungen

$$M = k \cdot \overline{M} \qquad k = n^1$$



$$M = k \cdot \overline{M} \qquad k = n^2$$

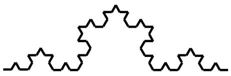


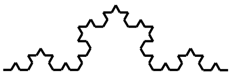
$$M = k \cdot \overline{M} \qquad k = n^3$$

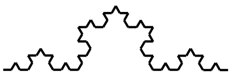
$$\boxed{k = n^{\dim(M)}}$$

$$k = n^{\dim(M)} \quad \Leftrightarrow \quad \log(k) = \dim(M) \cdot \log(n) \quad \Leftrightarrow \quad \boxed{\dim(M) = \frac{\log(k)}{\log(n)}}$$

Dimension der Schneeflockenkurve

 $\xrightarrow[n=3]{1:3}$  $k = 4$

 $\xrightarrow[n=3^2]{1:3^2}$ $k = 4^2$

 $\xrightarrow[n=3^3]{1:3^3}$ $k = 4^3$

$n=3^i$ $k=4^i$

$$\dim(\text{Sfk}) = \frac{\log(k)}{\log(n)} = \frac{\log(4^i)}{\log(3^i)} = \frac{i\cdot\log(4)}{i\cdot\log(3)} = \frac{\log(4)}{\log(3)} = 1{,}261859507$$

Figuren mit gebrochener Dimension heißen **Fraktale** !