Volumina und Oberflächen n-dimensionaler Kugeln

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Was bekannt ist :

	Radius	Volumen	Oberfläche
1-dimensionale Kugel			
2-dimensionale Kugel			
3-dimensionale Kugel K_3	R	$V_3 = \frac{4}{3}\pi R^3$	$O_3 = 4\pi R^2$
4-dimensionale Kugel			
5-dimensionale Kugel			
6-dimensionale Kugel			
.			
.			
.			
n-dimensionale Kugel			

Was bekannt ist :

		Radius	Volumen	Oberfläche
1-dimensionale Kugel	K_1	R	$V_1 = 2R$	$O_1 = 2$
2-dimensionale Kugel	K_2	R	$V_2 = \pi R^2$	$O_2 = 2\pi R$
3-dimensionale Kugel	K_3	R	$V_3 = \frac{4}{3}\pi R^3$	$O_3 = 4\pi R^2$

4-dimensionale Kugel

5-dimensionale Kugel

6-dimensionale Kugel

.
.
.

n-dimensionale Kugel

Bemerkung :
Dass $O_1 = 2$ und nicht $= 0$ ist, wird sich noch zeigen !

Elementare Forderungen, Gleichungen und Zusammenhänge :

(1)

$$K_1 = \left\{ P\left(x_1\right) \in \mathbb{R}^1 \;\middle|\; x_1^2 = R^2 \right\}$$

$$K_2 = \left\{ P\left(x_1 | x_2\right) \in \mathbb{R}^2 \;\middle|\; x_1^2 + x_2^2 = R^2 \right\}$$

$$K_3 = \left\{ P\left(x_1 | x_2 | x_3\right) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 = R^2 \right\}$$

$$\vdots$$

$$K_n = \left\{ P\left(x_1 | \ldots | x_n\right) \in \mathbb{R}^n \;\middle|\; x_1^2 \ldots + x_n^2 = R^2 \right\}$$

(2)

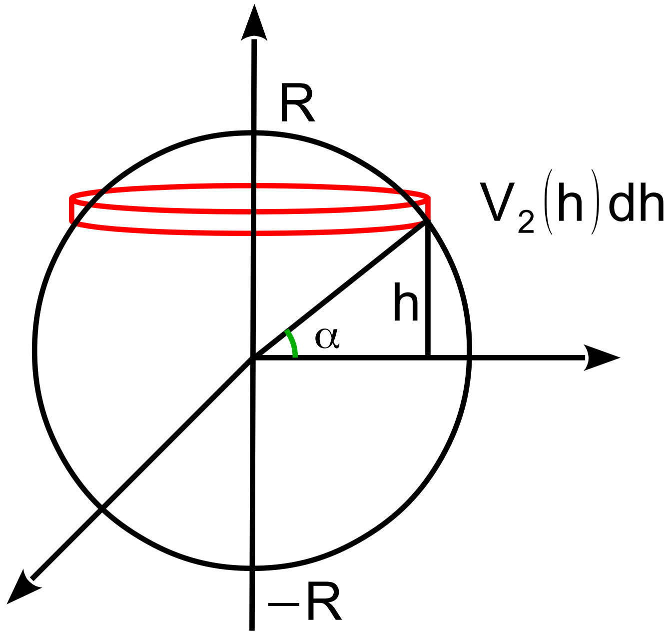
$$V_n \sim R^n \quad , \quad V_n = C_n R^n \quad , \quad V_1 = 2R \quad , \quad \boxed{C_1 = 2}$$

$$V_2 = \pi R^2 \quad , \quad \boxed{C_2 = \pi}$$

(3) $$V_n = \int_{-R}^{R} V_{n-1}(h)\ dh$$

$$V_3 = \int_{-R}^{R} V_2(h)\ dh$$

$$h = R\sin(\alpha)$$
$$dh = R\cos(\alpha)\,d\alpha$$
$$-R \;\hat{=}\; -\frac{\pi}{2} \qquad R \;\hat{=}\; \frac{\pi}{2}$$



$$V_n = \int_{-R}^{R} V_{n-1}(h)\ dh$$

$$V_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V_{n-1}(R\cos(\alpha))\,R\cos(\alpha)\ d\alpha$$

$$V_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} C_{n-1}(R\cos(\alpha))^{n-1}\,R\cos(\alpha)\ d\alpha$$

$$V_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} C_{n-1} \left(R \cos(\alpha) \right)^{n-1} R \cos(\alpha) \, d\alpha$$

$$V_n = \underbrace{C_{n-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha) \, d\alpha}_{C_n} \; R^n$$

$$\boxed{V_n = C_n R^n} \quad \text{mit} \quad \boxed{C_n = C_{n-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha) \, d\alpha} \quad \text{Rekursionsformel}$$

Rekursive Formeln für die Integrale $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\, d\alpha$:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\, d\alpha = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos^{n-1}(\alpha)}_{v}\, \underbrace{\cos(\alpha)}_{u'}\, d\alpha$$

$$u' = \cos(\alpha) \qquad v = \cos^{n-1}(\alpha)$$

$$u = \sin(\alpha) \qquad v' = (n-1)\cos^{n-2}(\alpha)\,(-\sin(\alpha))$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\, d\alpha = \left[\, uv \,\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u v'\, d\alpha$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\, d\alpha = \left[\, \sin(\alpha)\cos^{n-1}(\alpha) \,\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\alpha)\,(n-1)\, \cos^{n-2}(\alpha)\,(-\sin(\alpha))\, d\alpha$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\ d\alpha = \left[\ \sin(\alpha)\cos^{n-1}(\alpha)\ \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\alpha)\left(n-1\right)\cos^{n-2}(\alpha)\left(-\sin(\alpha)\right)\ d\alpha$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\ d\alpha = 0 + \left(n-1\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-2}(\alpha)\sin^2(\alpha)\ d\alpha$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\ d\alpha = \left(n-1\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-2}(\alpha)\left(1-\cos^2(\alpha)\right)\ d\alpha$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\ d\alpha = \left(n-1\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-2}(\alpha) - \cos^n(\alpha)\ d\alpha$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\ d\alpha = \left(n-1\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-2}(\alpha) - \left(n-1\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\ d\alpha$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\; d\alpha \;=\; (n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-2}(\alpha) \;-\; (n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\; d\alpha$$

$$n\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\; d\alpha \;=\; (n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-2}(\alpha)$$

$$\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\; d\alpha \;=\; \frac{n-1}{n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-2}(\alpha)}$$

Rekursionsformel

$$\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\ d\alpha = \frac{n-1}{n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-2}(\alpha)}$$

$$n = 2 \quad : \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\alpha)\ d\alpha = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^0(\alpha)$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\alpha)\ d\alpha = \frac{1}{2} \left[\ \alpha\ \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$\underline{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\alpha)\ d\alpha = \frac{1}{2}\,\pi}$$

$$\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\ d\alpha = \frac{n-1}{n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-2}(\alpha)}$$

$$n = 4 \quad : \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\alpha)\ d\alpha = \frac{3}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\alpha)$$

$$\underline{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\alpha)\ d\alpha = \frac{3}{4}\,\frac{1}{2}\,\pi}$$

$$n = 6 \quad : \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6(\alpha)\ d\alpha = \frac{5}{6} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\alpha)$$

$$\underline{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6(\alpha)\ d\alpha = \frac{5}{6}\,\frac{3}{4}\,\frac{1}{2}\,\pi}$$

Explizite Formeln für die Integrale $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(\alpha)\, d\alpha$ **:**

$n = g = \text{gerade}$:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^g(\alpha)\, d\alpha = \frac{g-1}{g}\,\frac{g-3}{g-2}\,\ldots\,\frac{3}{4}\,\frac{1}{2}\,\pi$$

Analog

$n = u = \text{ungerade}$:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^u(\alpha)\, d\alpha = \frac{u-1}{u}\,\frac{u-3}{u-2}\,\ldots\,\frac{4}{5}\,\frac{2}{3}\,2$$

Rekursive Berechnung der Volumina V_n :

$$V_1 = C_1 R = 2 R \qquad \boxed{C_1 = 2}$$

$$V_2 = C_2 R^2 = \pi R^2 \qquad \boxed{C_2 = \pi}$$

$$V_3 = C_3 R^2 \qquad C_3 = C_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3(\alpha)$$

$$C_3 = \pi \frac{2}{3} 2$$

$$\boxed{C_3 = \frac{4}{3} \pi} \qquad \boxed{V_3 = \frac{4}{3} \pi R^3}$$

$$V_4 = C_4 R^4$$

$$C_4 = C_3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\alpha)$$

$$C_4 = \frac{4}{3}\pi \frac{3}{4}\frac{1}{2}\pi$$

$$\boxed{C_4 = \frac{1}{2}\pi^2}$$

$$\boxed{V_4 = \frac{1}{2}\pi^2 R^4}$$

$$V_5 = C_5 R^5$$

$$C_5 = C_4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5(\alpha)$$

$$C_5 = \frac{1}{2}\pi^2 \frac{4}{5}\frac{2}{3} 2$$

$$\boxed{C_5 = \frac{8}{15}\pi^2}$$

$$\boxed{V_5 = \frac{8}{15}\pi^2 R^5}$$

$$V_6 = C_6 R^6$$

$$C_6 = C_5 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6(\alpha)$$

$$C_6 = \frac{8}{15} \pi^2 \frac{5}{6} \frac{3}{4} \frac{1}{2} \pi$$

$$\boxed{C_6 = \frac{1}{6} \pi^3}$$

$$\boxed{V_6 = \frac{1}{6} \pi^3 R^6}$$

$$V_7 = C_7 R^7$$

$$C_7 = C_6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^7(\alpha)$$

$$C_7 = \frac{1}{6} \pi^3 \frac{6}{7} \frac{4}{5} \frac{2}{3} 2$$

$$\boxed{C_7 = \frac{16}{105} \pi^3}$$

$$\boxed{V_7 = \frac{16}{105} \pi^3 R^7}$$

$$V_8 = C_8 R^8$$

$$C_8 = C_7 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^8(\alpha)$$

$$C_8 = \frac{16}{105} \pi^3 \frac{7}{8} \frac{5}{6} \frac{3}{4} \frac{1}{2} \pi$$

$$\boxed{C_8 = \frac{1}{24} \pi^4}$$

$$\boxed{V_8 = \frac{1}{24} \pi^4 R^8}$$

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Und so weiter !

Explizite Formeln der Volumina V_n für $n = g =$ gerade :

$n = g =$ gerade :

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{g}(\alpha)\, d\alpha = \frac{g-1}{g}\,\frac{g-3}{g-2}\,\ldots\,\frac{3}{4}\,\frac{1}{2}\,\pi$$

$n = u =$ ungerade :

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{u}(\alpha)\, d\alpha = \frac{u-1}{u}\,\frac{u-3}{u-2}\,\ldots\,\frac{4}{5}\,\frac{2}{3}\,2$$

$$C_g = C_{g-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{g}(\alpha)\, d\alpha$$

$$C_g = C_{g-2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{g-1}(\alpha)\, d\alpha \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{g}(\alpha)\, d\alpha$$

$$C_g = C_{g-2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{g-1}(\alpha)\, d\alpha \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{g}(\alpha)\, d\alpha$$

$$C_g = C_{g-2}\, \frac{g-2}{g-1}\,\frac{g-4}{g-3}\,\ldots\,\frac{4}{5}\,\frac{2}{3}\,2\;\; \frac{g-1}{g}\,\frac{g-3}{g-2}\,\ldots\,\frac{3}{4}\,\frac{1}{2}\,\pi$$

$$C_g = C_{g-2} \frac{g-2}{g-1} \frac{g-4}{g-3} \cdots \frac{4}{5} \frac{2}{3} 2 \quad \frac{g-1}{g} \frac{g-3}{g-2} \cdots \frac{3}{4} \frac{1}{2} \pi$$

$$C_g = C_{g-2} \frac{g-1}{g} \frac{g-2}{g-1} \frac{g-3}{g-2} \frac{g-4}{g-3} \cdots \frac{4}{5} \frac{3}{4} \frac{2}{3} \frac{1}{2} 2\pi$$

$$\boxed{C_g = C_{g-2} \frac{1}{g} 2\pi}$$

$$\Rightarrow \quad C_g = C_{g-4} \frac{1}{g-2} 2\pi \quad \frac{1}{g} 2\pi$$

$$\Rightarrow \quad C_g = C_{g-6} \frac{1}{g-4} 2\pi \quad \frac{1}{g-2} 2\pi \quad \frac{1}{g} 2\pi$$

$$\Rightarrow \quad C_g = C_2 \frac{1}{4} 2\pi \quad \cdots \quad \frac{1}{g-2} 2\pi \quad \frac{1}{g} 2\pi \qquad , \qquad \boxed{C_2 = \pi}$$

$$\Rightarrow \quad C_g = \pi \frac{1}{4} 2\pi \quad \cdots \quad \frac{1}{g-2} 2\pi \quad \frac{1}{g} 2\pi$$

$$\Rightarrow \quad C_g = \frac{1}{2} 2\pi \quad \frac{1}{4} 2\pi \quad \cdots \quad \frac{1}{g-2} 2\pi \quad \frac{1}{g} 2\pi$$

$$\Rightarrow \quad C_g = \frac{1}{2 \cdot 4 \cdot \ldots \cdot (g-2) \cdot g} (2\pi)^{\frac{g}{2}}$$

$$\Rightarrow \quad \boxed{C_g = \frac{1}{g!!} (2\pi)^{\frac{g}{2}}} \quad \Rightarrow \quad \boxed{V_g = \frac{1}{g!!} (2\pi)^{\frac{g}{2}} R^g}$$

Explizite Formeln der Volumina V_n für $n = u = \text{ungerade}$:

$n = g = \text{gerade}$:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^g(\alpha)\, d\alpha = \frac{g-1}{g} \frac{g-3}{g-2} \dots \frac{3}{4} \frac{1}{2} \pi$$

$n = u = \text{ungerade}$:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^u(\alpha)\, d\alpha = \frac{u-1}{u} \frac{u-3}{u-2} \dots \frac{4}{5} \frac{2}{3} 2$$

$$C_u = C_{u-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^u(\alpha)\, d\alpha$$

$$C_u = C_{u-2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{u-1}(\alpha)\, d\alpha \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^u(\alpha)\, d\alpha$$

$$C_u = C_{u-2} \frac{u-2}{u-1} \frac{u-4}{u-3} \dots \frac{3}{4} \frac{1}{2} \pi \; \frac{u-1}{u} \frac{u-3}{u-2} \dots \frac{4}{5} \frac{2}{3} 2$$

$$C_u = C_{u-2} \frac{u-1}{u} \frac{u-2}{u-1} \frac{u-3}{u-2} \frac{u-4}{u-3} \dots \frac{4}{5} \frac{3}{4} \frac{2}{3} \frac{1}{2} \, 2\pi$$

$$C_u = C_{u-2} \; \frac{u-1}{u} \frac{u-2}{u-1} \frac{u-3}{u-2} \frac{u-4}{u-3} \; \dots \; \frac{4}{5} \frac{3}{4} \frac{2}{3} \frac{1}{2} \; 2\pi$$

$$\boxed{C_u = C_{u-2} \; \frac{1}{u} \; 2\pi}$$

$$\Rightarrow \quad C_u = C_{u-4} \; \frac{1}{u-2} 2\pi \; \frac{1}{u} 2\pi$$

$$\Rightarrow \quad C_u = C_{u-6} \; \frac{1}{u-4} 2\pi \; \frac{1}{u-2} 2\pi \; \frac{1}{u} 2\pi$$

$$\Rightarrow \quad C_u = C_1 \; \frac{1}{3} 2\pi \; \dots \; \frac{1}{u-2} 2\pi \; \frac{1}{u} 2\pi \quad , \qquad \boxed{C_1 = 2}$$

$$\Rightarrow \quad C_u = 2 \; \frac{1}{3} 2\pi \; \dots \; \frac{1}{u-2} 2\pi \; \frac{1}{u} 2\pi$$

$$\Rightarrow \quad C_u = \frac{1}{3 \cdot 5 \cdot \dots \cdot (u-2) \cdot u} \; 2 \; (2\pi)^{\frac{u-1}{2}}$$

$$\Rightarrow \quad C_u = \frac{1}{3 \cdot 5 \cdot \dots \cdot (u-2) \cdot u} \; 2 \; (2\pi)^{\frac{u-1}{2}}$$

$$\Rightarrow \quad C_u = \frac{1}{3 \cdot 5 \cdot \ldots \cdot (u-2) \cdot u} \; 2 \; (2\pi)^{\frac{u-1}{2}}$$

$$\Rightarrow \quad \boxed{C_u = \frac{1}{u!!} \; 2 \; (2\pi)^{\frac{u-1}{2}}} \quad \Rightarrow \quad \boxed{V_u = \frac{1}{u!!} \; 2 \; (2\pi)^{\frac{u-1}{2}} \; R^u}$$

Die Volumina von n-dimensionalen Kugeln sind also gegeben durch :

$$\boxed{V_n = \left\{ \begin{array}{ll} \frac{1}{n!!} \; (2\pi)^{\frac{n}{2}} \; R^n & \text{falls } n \text{ gerade} \\ \frac{1}{n!!} \; 2 \; (2\pi)^{\frac{n-1}{2}} \; R^n & \text{falls } n \text{ ungerade} \end{array} \right\}}$$

Explizite Formeln der Oberflächen O_n :

$$V_n = \int_0^R O_n(r)\ dr$$

$$O_n = \frac{d}{dR} V_n$$

Die Oberflächen von n-dimensionalen Kugeln sind gegeben durch :

$$\boxed{O_n = \begin{cases} \dfrac{n}{n!!}\ (2\pi)^{\frac{n}{2}}\ R^{n-1} & \text{falls } n \text{ gerade} \\[2ex] \dfrac{n}{n!!}\ 2\ (2\pi)^{\frac{n-1}{2}}\ R^{n-1} & \text{falls } n \text{ ungerade} \end{cases}}$$

Volumen und Oberflächen der n-dimensionen Einheitskugeln

Volumen V_n , R=1	Oberfläche O_n , R=1
$V_1 = 2 \approx 2$	$O_1 = 2$
$V_2 = \pi \approx 3{,}14$	$O_2 = 2\pi \approx 6{,}28$
$V_3 = \frac{4}{3}\pi \approx 4{,}19$	$O_3 = 4\pi \approx 12{,}57$
$V_4 = \frac{1}{2}\pi^2 \approx 4{,}93$	$O_4 = 2\pi^2 \approx 19{,}74$
$V_5 = \frac{8}{15}\pi^2 \approx 5{,}26$	$O_5 = \frac{8}{3}\pi^2 \approx 26{,}31$
$V_6 = \frac{1}{6}\pi^3 \approx 5{,}17$	$O_6 = \pi^3 \approx 31{,}01$
$V_7 = \frac{16}{105}\pi^3 \approx 4{,}72$	$O_7 = \frac{16}{15}\pi^3 \approx 33{,}07$
$V_8 = \frac{1}{24}\pi^4 \approx 4{,}06$	$V_8 = \frac{1}{3}\pi^4 \approx 32{,}47$

Diagramm für Volumen und Oberfläche der n-dimensionalen Einheitskugeln