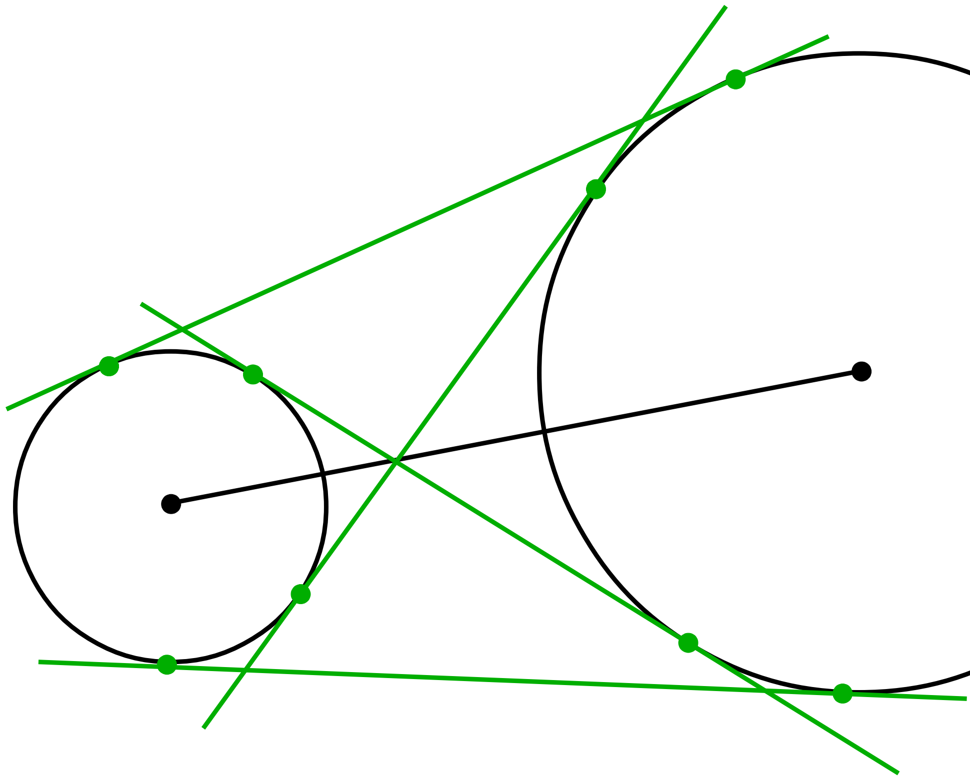
Gemeinsame Tangenten zweier Kreise

Arno Fehringer , Gymnasiallehrer für Mathematik und Physik

November 2023

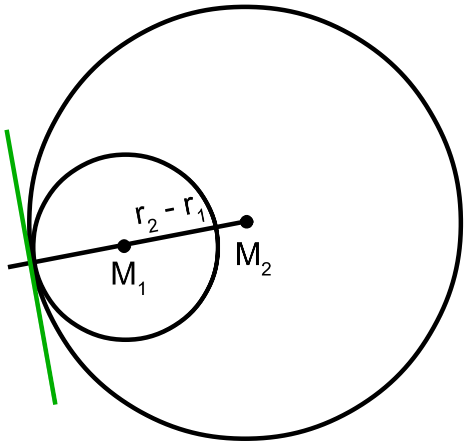
Aufgabe :

Zu zwei gegebenen Kreisen $K_{M_1 r_1}$, $K_{M_2 r_2}$ mit $r_1 < r_2$ sollen die Berührpunkte aller gemeinsamen Tangenten bestimmt werden !



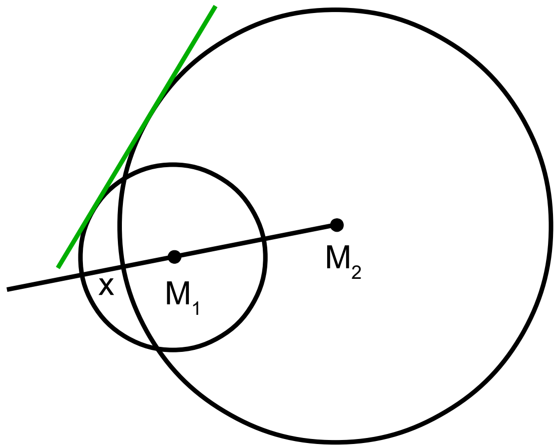
Fallunterscheidungen :

I.



$$r_2 - r_1 = \left|\overrightarrow{M_1M_2}\right| < r_2$$

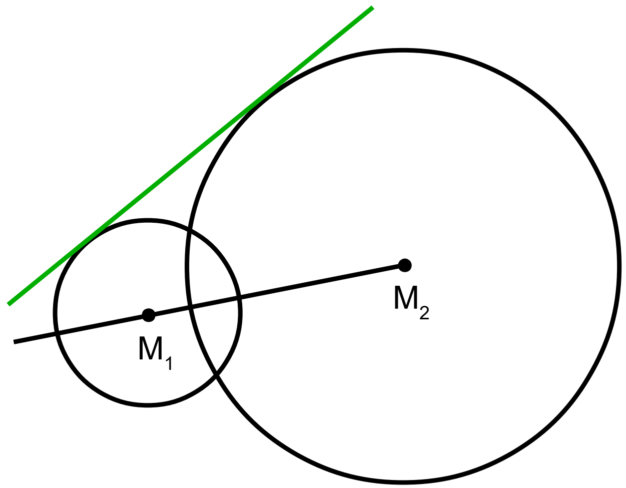
II.



$$r_2 < r_1 + \left|\overrightarrow{M_1M_2}\right|$$

$$r_2 - r_1 < \left|\overrightarrow{M_1M_2}\right| < r_2$$

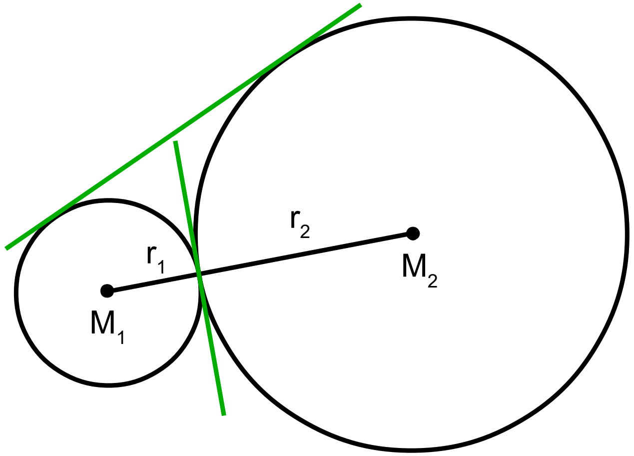
III.



$$r_2 \leq \left|\overrightarrow{M_1M_2}\right| < r_2 + r_1$$

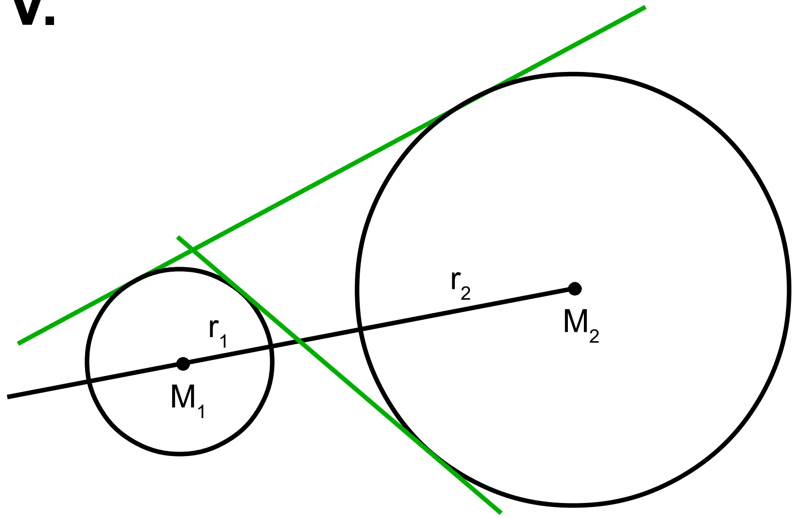
Fallunterscheidungen :

IV.



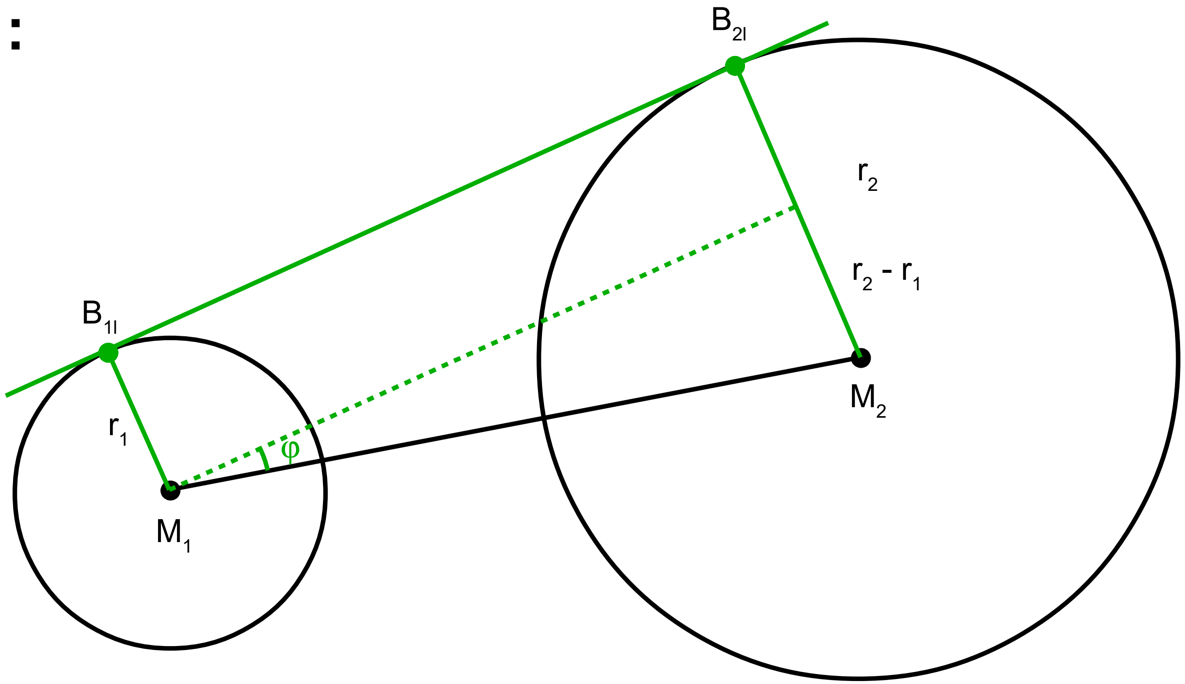
$$r_2 + r_1 = \left|\overrightarrow{M_1M_2}\right|$$

V.



$$r_2 + r_1 < \left|\overrightarrow{M_1M_2}\right|$$

Zu I. , II. , III. :



$$\sin(\varphi) = \frac{r_2 - r_1}{\left|\overrightarrow{M_1 M_2}\right|} = \frac{r_2 - r_1}{D} \qquad \text{mit} \qquad \boxed{D := \left|\overrightarrow{M_1 M_2}\right|}$$

$$\cos(\varphi) = \sqrt{1 - \sin^2(\varphi)} = \sqrt{1 - \frac{(r_2 - r_1)^2}{D^2}} = \frac{\sqrt{D^2 - (r_2 - r_1)^2}}{D}$$

Also

$$\boxed{\sin(\varphi) = \frac{r_2 - r_1}{D}}$$

$$\boxed{D := \left|\overrightarrow{M_1 M_2}\right|}$$

$$\boxed{\cos(\varphi) = \frac{\sqrt{D^2 - (r_2 - r_1)^2}}{D}}$$

Darstellung der „linken“ Berührpunkte B_{1l} , B_{2l} **:**

Setze : $\vec{u} := \dfrac{\overrightarrow{M_1 M_2}}{D}$ Einheitsvektor von M_1 nach M_2

$$B_{1l}: \quad \begin{pmatrix} x_{B_{1l}} \\ y_{B_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} \cos(\varphi + 90°) & -\sin(\varphi + 90°) \\ \sin(\varphi + 90°) & \cos(\varphi + 90°) \end{bmatrix} \vec{u}$$

$$B_{2l}: \quad \begin{pmatrix} x_{B_{2l}} \\ y_{B_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} \cos(\varphi + 90°) & -\sin(\varphi + 90°) \\ \sin(\varphi + 90°) & \cos(\varphi + 90°) \end{bmatrix} \vec{u}$$

$$B_{1l}: \quad \begin{pmatrix} x_{B_{1l}} \\ y_{B_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} -\sin(\varphi) & -\cos(\varphi) \\ \cos(\varphi) & -\sin(\varphi) \end{bmatrix} \vec{u}$$

$$B_{2l}: \quad \begin{pmatrix} x_{B_{2l}} \\ y_{B_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} -\sin(\varphi) & -\cos(\varphi) \\ \cos(\varphi) & -\sin(\varphi) \end{bmatrix} \vec{u}$$

$$B_{1l}: \quad \begin{pmatrix} x_{B_{1l}} \\ y_{B_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} -\dfrac{r_2-r_1}{D} & -\dfrac{\sqrt{D^2-(r_2-r_1)^2}}{D} \\ \dfrac{\sqrt{D^2-(r_2-r_1)^2}}{D} & -\dfrac{r_2-r_1}{D} \end{bmatrix} \dfrac{\overrightarrow{M_1M_2}}{D}$$

$$B_{2l}: \quad \begin{pmatrix} x_{B_{2l}} \\ y_{B_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} -\dfrac{r_2-r_1}{D} & -\dfrac{\sqrt{D^2-(r_2-r_1)^2}}{D} \\ \dfrac{\sqrt{D^2-(r_2-r_1)^2}}{D} & -\dfrac{r_2-r_1}{D} \end{bmatrix} \dfrac{\overrightarrow{M_1M_2}}{D}$$

Im Weiteren folgen einige Vereinfachungen :

$$B_{1l}: \quad \begin{pmatrix} x_{B_{1l}} \\ y_{B_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2}\begin{bmatrix} -(r_2-r_1) & -\sqrt{D^2-(r_2-r_1)^2} \\ \sqrt{D^2-(r_2-r_1)^2} & -(r_2-r_1) \end{bmatrix}\overrightarrow{M_1M_2}$$

$$B_{2l}: \quad \begin{pmatrix} x_{B_{2l}} \\ y_{B_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2}\begin{bmatrix} -(r_2-r_1) & -\sqrt{D^2-(r_2-r_1)^2} \\ \sqrt{D^2-(r_2-r_1)^2} & -(r_2-r_1) \end{bmatrix}\overrightarrow{M_1M_2}$$

$$B_{1l}: \quad \boxed{\begin{pmatrix} x_{B_{1l}} \\ y_{B_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2}\begin{bmatrix} -(r_2-r_1) & -\sqrt{D^2-(r_2-r_1)^2} \\ \sqrt{D^2-(r_2-r_1)^2} & -(r_2-r_1) \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

$$B_{2l}: \quad \boxed{\begin{pmatrix} x_{B_{2l}} \\ y_{B_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2}\begin{bmatrix} -(r_2-r_1) & -\sqrt{D^2-(r_2-r_1)^2} \\ \sqrt{D^2-(r_2-r_1)^2} & -(r_2-r_1) \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

Darstellung der „rechten“ Berührpunkte B_{1r} **,** B_{2r} **:**

$$B_{1r}: \quad \begin{pmatrix} x_{B_{1r}} \\ y_{B_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} \cos\left(-\left(\varphi+90°\right)\right) & -\sin\left(-\left(\varphi+90°\right)\right) \\ \sin\left(-\left(\varphi+90°\right)\right) & \cos\left(-\left(\varphi+90°\right)\right) \end{bmatrix} \vec{u}$$

$$B_{2r}: \quad \begin{pmatrix} x_{B_{2r}} \\ y_{B_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} \cos\left(-\left(\varphi+90°\right)\right) & -\sin\left(-\left(\varphi+90°\right)\right) \\ \sin\left(-\left(\varphi+90°\right)\right) & \cos\left(-\left(\varphi+90°\right)\right) \end{bmatrix} \vec{u}$$

$$B_{1r}: \quad \begin{pmatrix} x_{B_{1r}} \\ y_{B_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} \cos\left(\varphi+90°\right) & \sin\left(\varphi+90°\right) \\ -\sin\left(\varphi+90°\right) & \cos\left(\varphi+90°\right) \end{bmatrix} \vec{u}$$

$$B_{2r}: \quad \begin{pmatrix} x_{B_{2r}} \\ y_{B_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} \cos\left(\varphi+90°\right) & \sin\left(\varphi+90°\right) \\ -\sin\left(\varphi+90°\right) & \cos\left(\varphi+90°\right) \end{bmatrix} \vec{u}$$

$$B_{1r}: \quad \begin{pmatrix} x_{B_{1r}} \\ y_{B_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} \cos(\varphi+90°) & \sin(\varphi+90°) \\ -\sin(\varphi+90°) & \cos(\varphi+90°) \end{bmatrix} \vec{u}$$

$$B_{2r}: \quad \begin{pmatrix} x_{B_{2r}} \\ y_{B_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} \cos(\varphi+90°) & \sin(\varphi+90°) \\ -\sin(\varphi+90°) & \cos(\varphi+90°) \end{bmatrix} \vec{u}$$

$$B_{1r}: \quad \begin{pmatrix} x_{B_{1r}} \\ y_{B_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} -\sin(\varphi) & \cos(\varphi) \\ -\cos(\varphi) & -\sin(\varphi) \end{bmatrix} \vec{u}$$

$$B_{2r}: \quad \begin{pmatrix} x_{B_{2r}} \\ y_{B_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} -\sin(\varphi) & \cos(\varphi) \\ -\cos(\varphi) & -\sin(\varphi) \end{bmatrix} \vec{u}$$

Wegen $\sin(\varphi) = \dfrac{r_2 - r_1}{D} \quad , \quad \cos(\varphi) = \dfrac{\sqrt{D^2 - (r_2 - r_1)^2}}{D} \quad , \quad \vec{u} := \dfrac{\overrightarrow{M_1 M_2}}{D} \quad :$

$$B_{1r}\ :\quad \begin{pmatrix} x_{B_{1r}} \\ y_{B_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} -\sin(\varphi) & \cos(\varphi) \\ -\cos(\varphi) & -\sin(\varphi) \end{bmatrix} \vec{u}$$

$$B_{2r}\ :\quad \begin{pmatrix} x_{B_{2r}} \\ y_{B_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} -\sin(\varphi) & \cos(\varphi) \\ -\cos(\varphi) & -\sin(\varphi) \end{bmatrix} \vec{u}$$

$$B_{1r}\ :\quad \begin{pmatrix} x_{B_{1r}} \\ y_{B_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} -\dfrac{r_2-r_1}{D} & \dfrac{\sqrt{D^2 - \left(r_2-r_1\right)^2}}{D} \\ -\dfrac{\sqrt{D^2 - \left(r_2-r_1\right)^2}}{D} & -\dfrac{r_2-r_1}{D} \end{bmatrix} \dfrac{\overrightarrow{M_1 M_2}}{D}$$

$$B_{2r}\ :\quad \begin{pmatrix} x_{B_{2r}} \\ y_{B_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} -\dfrac{r_2-r_1}{D} & \dfrac{\sqrt{D^2 - \left(r_2-r_1\right)^2}}{D} \\ -\dfrac{\sqrt{D^2 - \left(r_2-r_1\right)^2}}{D} & -\dfrac{r_2-r_1}{D} \end{bmatrix} \dfrac{\overrightarrow{M_1 M_2}}{D}$$

$$B_{1r}: \quad \begin{pmatrix} x_{B_{1r}} \\ y_{B_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2} \begin{bmatrix} -\left(r_2 - r_1\right) & \sqrt{D^2 - \left(r_2 - r_1\right)^2} \\ -\sqrt{D^2 - \left(r_2 - r_1\right)^2} & -\dfrac{r_2 - r_1}{D} \end{bmatrix} \overrightarrow{M_1 M_2}$$

$$B_{2r}: \quad \begin{pmatrix} x_{B_{2r}} \\ y_{B_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2} \begin{bmatrix} -\left(r_2 - r_1\right) & \sqrt{D^2 - \left(r_2 - r_1\right)^2} \\ -\sqrt{D^2 - \left(r_2 - r_1\right)^2} & -\left(r_2 - r_1\right) \end{bmatrix} \overrightarrow{M_1 M_2}$$

$$B_{1r}: \quad \boxed{\begin{pmatrix} x_{B_{1r}} \\ y_{B_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2} \begin{bmatrix} -\left(r_2 - r_1\right) & \sqrt{D^2 - \left(r_2 - r_1\right)^2} \\ -\sqrt{D^2 - \left(r_2 - r_1\right)^2} & -\left(r_2 - r_1\right) \end{bmatrix} \begin{pmatrix} x_{M_2} - x_{M_1} \\ y_{M_2} - y_{M_1} \end{pmatrix}}$$

$$B_{2r}: \quad \boxed{\begin{pmatrix} x_{B_{2r}} \\ y_{B_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2} \begin{bmatrix} -\left(r_2 - r_1\right) & \sqrt{D^2 - \left(r_2 - r_1\right)^2} \\ -\sqrt{D^2 - \left(r_2 - r_1\right)^2} & -\left(r_2 - r_1\right) \end{bmatrix} \begin{pmatrix} x_{M_2} - x_{M_1} \\ y_{M_2} - y_{M_1} \end{pmatrix}}$$

Zusammenfassende Darstellung der Berührpunktepaare B_{1l} , B_{2l} und B_{1r} , B_{2r} zu I. , II. , III. :

B_{1l} :

$$\boxed{\begin{pmatrix} x_{B_{1l}} \\ y_{B_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2}\begin{bmatrix} -\left(r_2-r_1\right) & -\sqrt{D^2 - \left(r_2-r_1\right)^2} \\ \sqrt{D^2 - \left(r_2-r_1\right)^2} & -\left(r_2-r_1\right) \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

B_{2l} :

$$\boxed{\begin{pmatrix} x_{B_{2l}} \\ y_{B_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2}\begin{bmatrix} -\left(r_2-r_1\right) & -\sqrt{D^2 - \left(r_2-r_1\right)^2} \\ \sqrt{D^2 - \left(r_2-r_1\right)^2} & -\left(r_2-r_1\right) \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

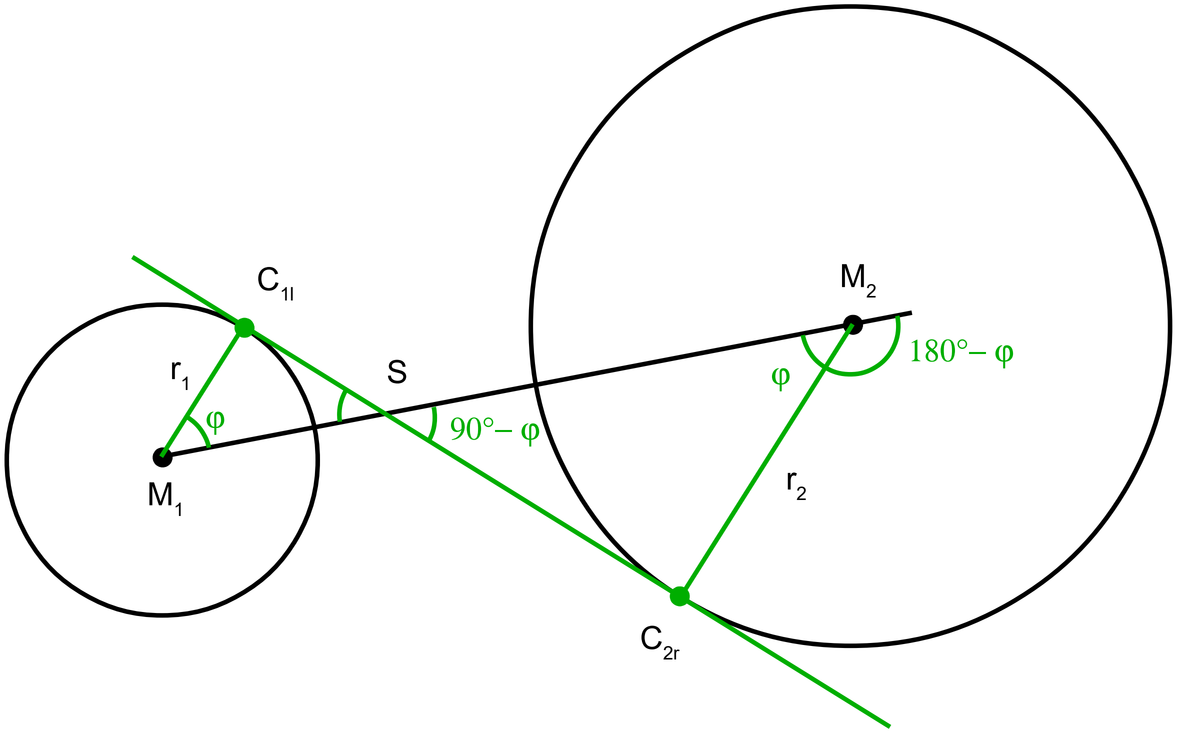
B_{1r} :

$$\boxed{\begin{pmatrix} x_{B_{1r}} \\ y_{B_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2}\begin{bmatrix} -\left(r_2-r_1\right) & \sqrt{D^2 - \left(r_2-r_1\right)^2} \\ -\sqrt{D^2 - \left(r_2-r_1\right)^2} & -\left(r_2-r_1\right) \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

B_{2r} :

$$\boxed{\begin{pmatrix} x_{B_{2r}} \\ y_{B_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2}\begin{bmatrix} -\left(r_2-r_1\right) & \sqrt{D^2 - \left(r_2-r_1\right)^2} \\ -\sqrt{D^2 - \left(r_2-r_1\right)^2} & -\left(r_2-r_1\right) \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

Zu IV. , V. :



Nach dem **2. Strahlensatz** wird die Strecke $\overline{M_1 M_2}$ im Verhältnis $\dfrac{r_2}{r_1}$ aufgeteilt, das heißt $\left|\overrightarrow{M_1 S}\right| = \dfrac{r_1}{r_1+r_2}\left|\overrightarrow{M_1 M_2}\right|$, $\left|\overrightarrow{S M_2}\right| = \dfrac{r_2}{r_1+r_2}\left|\overrightarrow{M_1 M_2}\right|$.

$$\left|\overrightarrow{M_1 S}\right| = \frac{r_1}{r_1+r_2}\left|\overrightarrow{M_1 M_2}\right| \quad , \quad \left|\overrightarrow{S M_2}\right| = \frac{r_2}{r_1+r_2}\left|\overrightarrow{M_1 M_2}\right|$$

$$\cos(\varphi) = \frac{r_1}{\left|\overrightarrow{M_1 S}\right|} = \frac{r_1}{\frac{r_1}{r_1+r_2}\left|\overrightarrow{M_1 M_2}\right|} = \frac{r_1+r_2}{\left|\overrightarrow{M_1 M_2}\right|}$$

$$\cos(\varphi) = \frac{r_1+r_2}{\left|\overrightarrow{M_1 M_2}\right|}$$

$$\boxed{D := \left|\overrightarrow{M_1 M_2}\right|}$$

$$\boxed{\cos(\varphi) = \frac{r_1+r_2}{D}}$$

$$\boxed{\sin(\varphi) = \frac{\sqrt{D^2 - \left(r_1+r_2\right)^2}}{D}}$$

Darstellung der Berührpunkte C_{1l} , C_{2r} **:**

Setze : $\vec{u} := \dfrac{\overrightarrow{M_1 M_2}}{D}$ Einheitsvektor von M_1 nach M_2

$$C_{1l}: \quad \begin{pmatrix} x_{C_{1l}} \\ y_{C_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \vec{u}$$

$$C_{2r}: \quad \begin{pmatrix} x_{C_{2r}} \\ y_{C_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} \cos\left(-\left(180°-\varphi\right)\right) & -\sin\left(-\left(180°-\varphi\right)\right) \\ \sin\left(-\left(180°-\varphi\right)\right) & \cos\left(-\left(180°-\varphi\right)\right) \end{bmatrix} \vec{u}$$

$$C_{1l}: \quad \begin{pmatrix} x_{C_{1l}} \\ y_{C_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \vec{u}$$

$$C_{2r}: \quad \begin{pmatrix} x_{C_{2r}} \\ y_{C_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} \cos\left(180°-\varphi\right) & \sin\left(180°-\varphi\right) \\ -\sin\left(180°-\varphi\right) & \cos\left(180°-\varphi\right) \end{bmatrix} \vec{u}$$

$$C_{1l}: \quad \begin{pmatrix} x_{C_{1l}} \\ y_{C_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \vec{u}$$

$$C_{2r}: \quad \begin{pmatrix} x_{C_{2r}} \\ y_{C_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} \cos(180° - \varphi) & \sin(180° - \varphi) \\ -\sin(180° - \varphi) & \cos(180° - \varphi) \end{bmatrix} \vec{u}$$

$$C_{1l}: \quad \begin{pmatrix} x_{C_{1l}} \\ y_{C_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \vec{u}$$

$$C_{2r}: \quad \begin{pmatrix} x_{C_{2r}} \\ y_{C_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} -\cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & -\cos(\varphi) \end{bmatrix} \vec{u}$$

$$C_{1l}: \quad \begin{pmatrix} x_{C_{1l}} \\ y_{C_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2} \begin{bmatrix} r_1+r_2 & -\sqrt{D^2 - (r_1+r_2)^2} \\ \sqrt{D^2 - (r_1+r_2)^2} & r_1+r_2 \end{bmatrix} \overrightarrow{M_1 M_2}$$

$$C_{2r}: \quad \begin{pmatrix} x_{C_{2r}} \\ y_{C_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2} \begin{bmatrix} -(r_1+r_2) & \sqrt{D^2 - (r_1+r_2)^2} \\ -\sqrt{D^2 - (r_1+r_2)^2} & -(r_1+r_2) \end{bmatrix} \overrightarrow{M_1 M_2}$$

$$C_{1l}: \quad \boxed{\begin{pmatrix} x_{C_{1l}} \\ y_{C_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2} \begin{bmatrix} r_1+r_2 & -\sqrt{D^2 - (r_1+r_2)^2} \\ \sqrt{D^2 - (r_1+r_2)^2} & r_1+r_2 \end{bmatrix} \begin{pmatrix} x_{M_2} - x_{M_1} \\ y_{M_2} - y_{M_1} \end{pmatrix}}$$

$$C_{2r}: \quad \boxed{\begin{pmatrix} x_{C_{2r}} \\ y_{C_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2} \begin{bmatrix} -(r_1+r_2) & \sqrt{D^2 - (r_1+r_2)^2} \\ -\sqrt{D^2 - (r_1+r_2)^2} & -(r_1+r_2) \end{bmatrix} \begin{pmatrix} x_{M_2} - x_{M_1} \\ y_{M_2} - y_{M_1} \end{pmatrix}}$$

Darstellung der Berührpunkte C_{1r} **,** C_{2l} **:**

Setze : $\vec{u} := \dfrac{\overrightarrow{M_1 M_2}}{D}$ Einheitsvektor von M_1 nach M_2

$$C_{1r}: \quad \begin{pmatrix} x_{C_{1r}} \\ y_{C_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{bmatrix} \vec{u}$$

$$C_{2l}: \quad \begin{pmatrix} x_{C_{2l}} \\ y_{C_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} \cos\left(180°-\varphi\right) & -\sin\left(\left(180°-\varphi\right)\right) \\ \sin\left(\left(180°-\varphi\right)\right) & \cos\left(180°-\varphi\right) \end{bmatrix} \vec{u}$$

$$C_{1r}: \quad \begin{pmatrix} x_{C_{1r}} \\ y_{C_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix} \vec{u}$$

$$C_{2l}: \quad \begin{pmatrix} x_{C_{2l}} \\ y_{C_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} -\cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{bmatrix} \vec{u}$$

$$C_{1r}: \quad \begin{pmatrix} x_{C_{1r}} \\ y_{C_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + r_1 \begin{bmatrix} \dfrac{r_1+r_2}{D} & \dfrac{\sqrt{D^2 - \left(r_1+r_2\right)^2}}{D} \\ -\dfrac{\sqrt{D^2 - \left(r_1+r_2\right)^2}}{D} & \dfrac{r_1+r_2}{D} \end{bmatrix} \frac{\overrightarrow{M_1M_2}}{D}$$

$$C_{2l}: \quad \begin{pmatrix} x_{C_{2l}} \\ y_{C_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + r_2 \begin{bmatrix} -\dfrac{r_1+r_2}{D} & -\dfrac{\sqrt{D^2 - \left(r_1+r_2\right)^2}}{D} \\ \dfrac{\sqrt{D^2 - \left(r_1+r_2\right)^2}}{D} & -\dfrac{r_1+r_2}{D} \end{bmatrix} \frac{\overrightarrow{M_1M_2}}{D}$$

$$C_{1r}: \quad \begin{pmatrix} x_{C_{1r}} \\ y_{C_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2} \begin{bmatrix} r_1+r_2 & \sqrt{D^2 - \left(r_1+r_2\right)^2} \\ -\sqrt{D^2 - \left(r_1+r_2\right)^2} & r_1+r_2 \end{bmatrix} \overrightarrow{M_1M_2}$$

$$C_{2l}: \quad \begin{pmatrix} x_{C_{2l}} \\ y_{C_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2} \begin{bmatrix} -\left(r_1+r_2\right) & -\sqrt{D^2 - \left(r_1+r_2\right)^2} \\ \sqrt{D^2 - \left(r_1+r_2\right)^2} & -\left(r_1+r_2\right) \end{bmatrix} \overrightarrow{M_1M_2}$$

$$C_{1r}: \quad \boxed{\begin{pmatrix} x_{C_{1r}} \\ y_{C_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2}\begin{bmatrix} r_1+r_2 & \sqrt{D^2 - (r_1+r_2)^2} \\ -\sqrt{D^2 - (r_1+r_2)^2} & r_1+r_2 \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

$$C_{2l}: \quad \boxed{\begin{pmatrix} x_{C_{2l}} \\ y_{C_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2}\begin{bmatrix} -(r_1+r_2) & -\sqrt{D^2 - (r_1+r_2)^2} \\ \sqrt{D^2 - (r_1+r_2)^2} & -(r_1+r_2) \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

Bemerkung :

Die Darstellung der Berührpunktepaare B_{1l} **,** B_{2l} **bzw.** B_{1r} **,** B_{2r} **ist wie in den Fällen I. , II. , III. .**

Zusammenfassende Darstellung der Berührpunktepaare C_{1l} , C_{2r} und C_{1r} , C_{2l} zu den Fällen IV. , V. :

C_{1l} :

$$\boxed{\begin{pmatrix} x_{C_{1l}} \\ y_{C_{1l}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2}\begin{bmatrix} r_1+r_2 & -\sqrt{D^2 - (r_1+r_2)^2} \\ \sqrt{D^2 - (r_1+r_2)^2} & r_1+r_2 \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

C_{2r} :

$$\boxed{\begin{pmatrix} x_{C_{2r}} \\ y_{C_{2r}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2}\begin{bmatrix} -(r_1+r_2) & \sqrt{D^2 - (r_1+r_2)^2} \\ -\sqrt{D^2 -(r_1+r_2)^2} & -(r_1+r_2) \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

C_{1r} :

$$\boxed{\begin{pmatrix} x_{C_{1r}} \\ y_{C_{1r}} \end{pmatrix} = \begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} + \frac{r_1}{D^2}\begin{bmatrix} r_1+r_2 & \sqrt{D^2 - (r_1+r_2)^2} \\ -\sqrt{D^2 - (r_1+r_2)^2} & r_1+r_2 \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

C_{2l} :

$$\boxed{\begin{pmatrix} x_{C_{2l}} \\ y_{C_{2l}} \end{pmatrix} = \begin{pmatrix} x_{M_2} \\ y_{M_2} \end{pmatrix} + \frac{r_2}{D^2}\begin{bmatrix} -(r_1+r_2) & -\sqrt{D^2 - (r_1+r_2)^2} \\ \sqrt{D^2 - (r_1+r_2)^2} & -(r_1+r_2) \end{bmatrix}\begin{pmatrix} x_{M_2}-x_{M_1} \\ y_{M_2}-y_{M_1} \end{pmatrix}}$$

Anhang :

Symmetrien der Trigonometrischen Funktionen :

$$\sin(-\alpha) = -\sin(\alpha)$$
$$\cos(-\alpha) = \cos(\alpha)$$

Additionstheoreme :

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\sin(\varphi+90°) = \sin(\varphi)\cos(90°) + \cos(\varphi)\sin(90°) = \cos(\varphi)$$
$$\cos(\varphi+90°) = \cos(\varphi)\cos(90°) - \sin(\varphi)\sin(90°) = -\sin(\varphi)$$

$$\sin(-(\varphi+90°)) = -(\sin(\varphi)\cos(90°) + \cos(\varphi)\sin(90°)) = -\cos(\varphi)$$
$$\cos(-(\varphi+90°)) = \cos(\varphi)\cos(90°) - \sin(\varphi)\sin(90°) = -\sin(\varphi)$$

$$\sin(90°-\varphi) = \sin(90°)\cos(\varphi) - \cos(90°)\sin(\varphi) = \cos(\varphi)$$
$$\cos(90°-\varphi) = \cos(90°)\cos(\varphi) + \sin(90°)\sin(\varphi) = \sin(\varphi)$$

$$\sin(-(90°-\varphi)) = -\sin(90°)\cos(\varphi) + \cos(90°)\sin(\varphi) = -\cos(\varphi)$$
$$\cos(-(90°-\varphi)) = \cos(90°)\cos(\varphi) + \sin(90°)\sin(\varphi) = \sin(\varphi)$$

$$\sin(180°-\varphi) = \sin(180°)\cos(\varphi) - \cos(180°)\sin(\varphi) = \sin(\varphi)$$
$$\cos(180°-\varphi) = \cos(180°)\cos(\varphi) + \sin(180°)\sin(\varphi) = -\cos(\varphi)$$

Drehung eines Vektors $\vec{u} = \begin{pmatrix} x_u \\ y_u \end{pmatrix}$ in mathematisch positiver Richtung um den Winkel φ :

$$\begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{pmatrix} x_u \\ y_u \end{pmatrix}$$